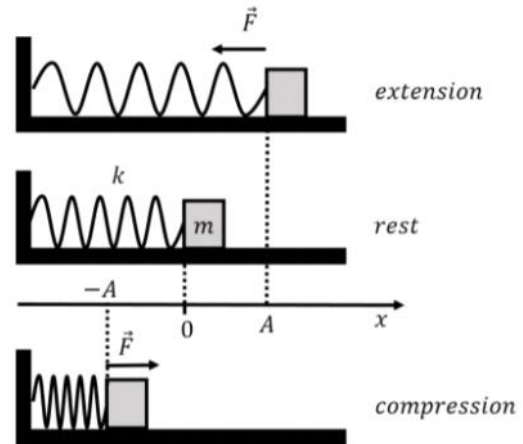


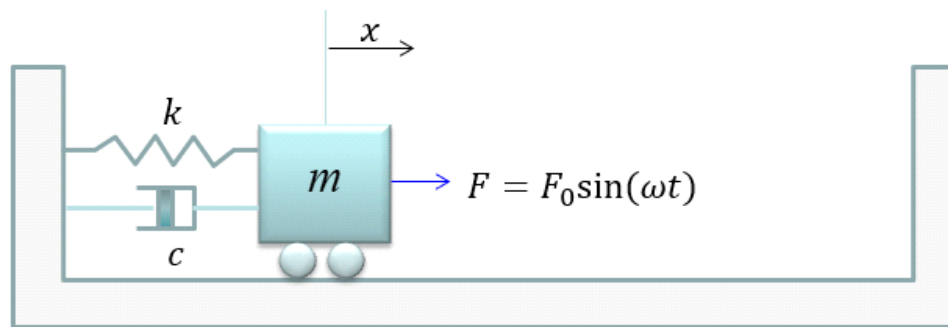
Section 3.6 Forced Oscillations and Resonance

In section 3.4 we studied the spring-dashpot system:

$$mx'' + cx' + kx = 0$$



Now that we have studied how to solve non-homogeneous systems we can broaden our horizons to systems that also have some external forcing.



$$mx'' + cx' + kx = F(t)$$

Essentially we imagine some external force acting on the mass attached to the spring on the dash pot. While $F(t)$ can be whatever we like, typically the most interesting cases for this external forcing are ones that are periodic in nature.

$$F(t) = F_0 \sin(\omega t)$$

$$F(t) = F_0 \cos(\omega t)$$

Section 3.6 Forced Oscillations and Resonance

Undamped Forced Oscillations

For now let's consider a system with no dash pot to damp oscillations forced with a force of $F = F_0 \cos(\omega t)$

$$mx'' + kx = F_0 \cos(\omega t)$$

We already know that to solve this system we need to find the complementary solution of coming from the homogeneous equation.

$$mx'' + kx = 0$$

For which we end up with general solution $x_c(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t)$, where $\omega_0 = \sqrt{\frac{k}{m}}$ and is called the natural frequency of oscillation. Let's just quickly see why that is again...

Now let's assume that $\omega \neq \omega_0$ and find our particular solution. Our equation is $mx'' + kx = F_0 \cos(\omega t)$ what should we guess for x_p ?

Let's find the undetermined coefficients.

Section 3.6 Forced Oscillations and Resonance

So we have the full general solution as:

$$x(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + \frac{F_0/m}{\omega_0^2 - \omega^2} \cos(\omega t)$$

Which we know we can re-write as

$$x(t) = C \cos(\omega_0 t - \alpha) + \frac{F_0/m}{\omega_0^2 - \omega^2} \cos(\omega t)$$

So that we can see that this is just a superposition of two cosine oscillations.

Let's suppose that $x(0) = 0$, $x'(0) = 0$ and find out constants in terms of m, k, F_0, ω

$$x(0) = c_1 \cos(\omega_0 0) + c_2 \sin(\omega_0 0) + \frac{F_0/m}{\omega_0^2 - \omega^2} \cos(\omega 0) = 0 \rightarrow c_1 + \frac{F_0/m}{\omega_0^2 - \omega^2} = 0 \rightarrow c_1 = -\frac{F_0/m}{\omega_0^2 - \omega^2}$$

and

$$x'(t) = -\omega_0 c_1 \sin(\omega_0 t) + \omega_0 c_2 \cos(\omega_0 t) - \omega \frac{F_0/m}{\omega_0^2 - \omega^2} \sin(\omega t)$$

so

$$x'(0) = -\omega_0 c_1 \sin(\omega_0 0) + \omega_0 c_2 \cos(\omega_0 0) - \omega \frac{F_0/m}{\omega_0^2 - \omega^2} \sin(\omega 0) \rightarrow \omega_0 c_2 = 0 \rightarrow c_2 = 0$$

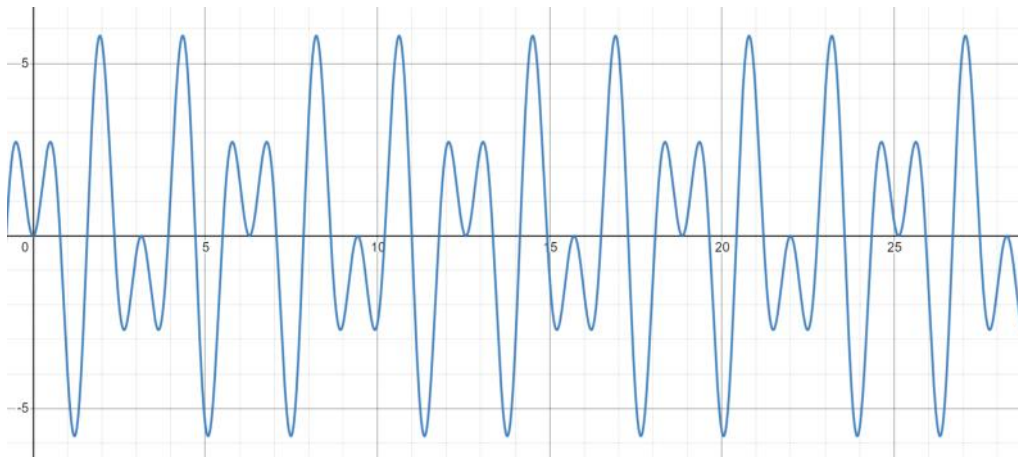
Thus:

$$x(t) = -\frac{F_0/m}{\omega_0^2 - \omega^2} \cos(\omega_0 t) + \frac{F_0/m}{\omega_0^2 - \omega^2} \cos(\omega t)$$

Or just

$$x(t) = \frac{F_0/m}{\omega_0^2 - \omega^2} (\cos(\omega t) - \cos(\omega_0 t))$$

Let's take $k = 9, m = 1, F_0 = 50, \omega = 5$ and plot.



For some fun with sliders: <https://www.desmos.com/calculator/vzsqr5mydp>
what happens when $\omega = \omega_0$?

Section 3.6 Forced Oscillations and Resonance

Before we dive into the case where $\omega = \omega_0$ let's look at the case where they are close to each other.

$$x(t) = \frac{F_0/m}{\omega_0^2 - \omega^2} (\cos(\omega t) - \cos(\omega_0 t))$$

To help us analytically understand the behavior of the system when the frequencies are close to each other we use the trig identity $2 \sin(a) \sin(b) = \cos(a - b) - \cos(a + b)$

If we take $a = \frac{1}{2}(\omega_0 - \omega)$ and $b = \frac{1}{2}(\omega_0 + \omega)$ the identity becomes...

So that we now have

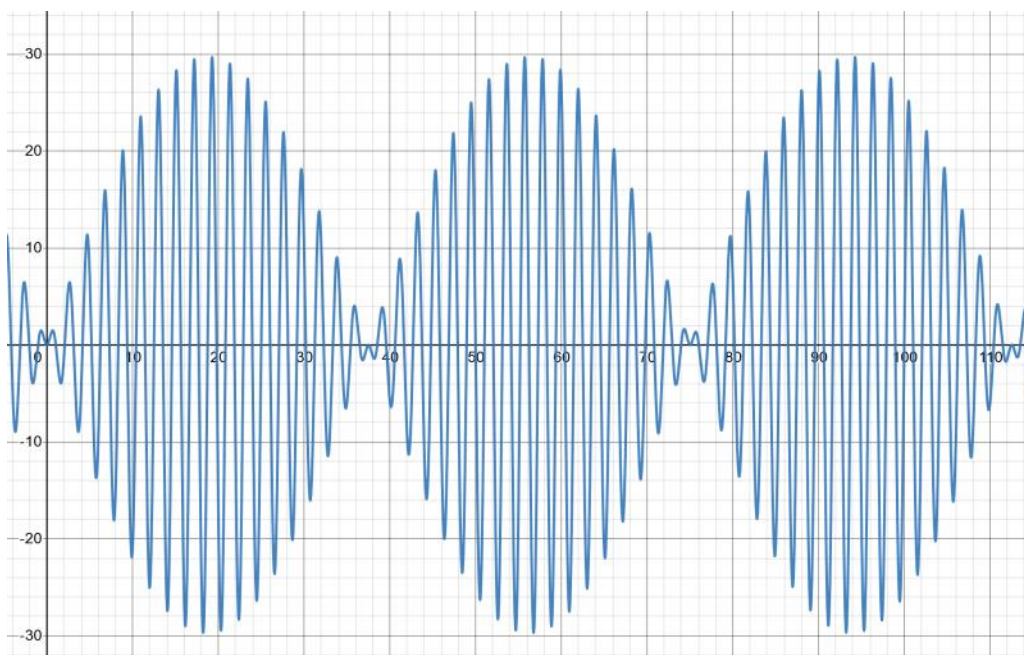
$$x(t) = \frac{F_0/m}{\omega_0^2 - \omega^2} (\cos(\omega t) - \cos(\omega_0 t)) = \frac{2F_0/m}{\omega_0^2 - \omega^2} \sin\left(\frac{1}{2}(\omega_0 - \omega)t\right) \sin\left(\frac{1}{2}(\omega_0 + \omega)t\right)$$

Note that when $\omega_0 \approx \omega$ that $|\omega_0 - \omega| \ll 1$ which means $\sin\left(\frac{1}{2}(\omega_0 - \omega)t\right)$ is a sin wave with a very very slow period compared to that of $\sin\left(\frac{1}{2}(\omega_0 + \omega)t\right)$. In this way we have a slowly changing amplitude in our solution ,

$$A(t) = \frac{2F_0/m}{\omega_0^2 - \omega^2} \sin\left(\frac{1}{2}(\omega_0 - \omega)t\right)$$

So we could look at our solution as

$$x(t) = A(t) \sin\left(\frac{1}{2}(\omega_0 + \omega)t\right)$$



Let's go play with our solution again and let $\omega \approx \omega_0$

<https://www.desmos.com/calculator/cw6f50eqqn>

Section 3.6 Forced Oscillations and Resonance

Let's look more closely at the case where $\omega = \omega_0$. This drastically changes our solution as now our previous guess for the particular solution $x_p = A\cos(\omega t) + B\sin(\omega t)$ would have $\omega = \omega_0$ and then,

$$x_p = A\cos(\omega_0 t) + B\sin(\omega_0 t),$$

would be a repeat of terms in the complementary solution

$$x_c(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t).$$

So according to our rules for the method of undetermined coefficients we would multiply our guess by the lowest power of t needed to prevent the repeat. In this case just one power. So we take x_p to be:

$$x_p = t(A\cos(\omega_0 t) + B\sin(\omega_0 t))$$

Making our full general solution out to be:

$$x(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + t(A\cos(\omega_0 t) + B\sin(\omega_0 t))$$

What will be the effect of this new term with multiple of t ? Let's find our undetermined coefficients.

$$x_p(t) = t(A\cos(\omega_0 t) + B\sin(\omega_0 t))$$

$$x_p'(t) = (A\cos(\omega_0 t) + B\sin(\omega_0 t)) + t(-\omega_0 A\sin(\omega_0 t) + \omega_0 B\cos(\omega_0 t))$$

$$x_p''(t) = (-\omega_0 A\sin(\omega_0 t) + \omega_0 B\cos(\omega_0 t)) + (-\omega_0 A\sin(\omega_0 t) + \omega_0 B\cos(\omega_0 t)) + t(-\omega_0^2 A\cos(\omega_0 t) - \omega_0^2 B\sin(\omega_0 t))$$

Then $m x'' + kx = F_0 \cos(\omega_0 t)$ gives us the equation

$$m \left((-\omega_0 A\sin(\omega_0 t) + \omega_0 B\cos(\omega_0 t)) + (-\omega_0 A\sin(\omega_0 t) + \omega_0 B\cos(\omega_0 t)) + t(-\omega_0^2 A\cos(\omega_0 t) - \omega_0^2 B\sin(\omega_0 t)) \right) + k(t(A\cos(\omega_0 t) + B\sin(\omega_0 t))) = F_0 \cos(\omega_0 t)$$

$$(2m\omega_0 B - t m \omega_0^2 A + t k A) \cos(\omega_0 t) + (-2m\omega_0 A - t m \omega_0^2 B + t k B) \sin(\omega_0 t) = F_0 \cos(\omega_0 t)$$

Now what is $m\omega_0^2 = ?$

So then we just have:

Which tells us that:

$$2m\omega_0 B = F_0$$

$$-2m\omega_0 A = 0$$

And so

$$x_p = \frac{F_0}{2m\omega_0} \sin(\omega_0 t)$$

Section 3.6 Forced Oscillations and Resonance

Let's again take $x(0) = x'(0) = 0$

$$x(t) = c_1 \cos(\omega_0 t) + c_2 \sin(\omega_0 t) + t \left(\frac{F_0}{2m\omega_0} \sin(\omega_0 t) \right)$$

$$x(0) = c_1 = 0$$

$$x'(t) = -\omega_0 c_1 \sin(\omega_0 t) + \omega_0 c_2 \cos(\omega_0 t) + \left(\frac{F_0}{2m\omega_0} \sin(\omega_0 t) \right) + t \left(\frac{\omega_0 F_0}{2m\omega_0} \cos(\omega_0 t) \right)$$

$$x'(0) = \omega_0 c_2 = 0$$

so

$$x(t) = t \left(\frac{F_0}{2m\omega_0} \sin(\omega_0 t) \right)$$

This is a case we call resonance. Here the forcing is going at the same frequency as the natural frequency of the spring, a lot like timing your swing pushes right to go higher and higher!

If you want to play with some sliders head over to:

<https://www.desmos.com/calculator/m23nnolvht>

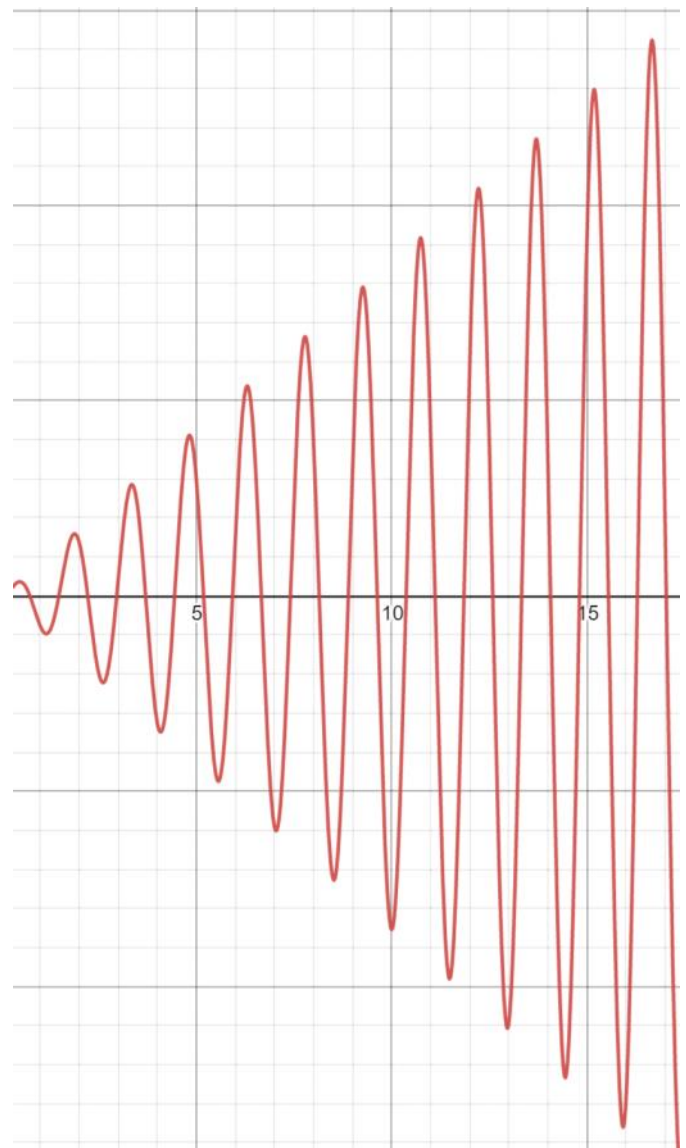
Resonance can be useful in some cases, like making lasers work but disastrous in others like the Tacoma bridge collapse.

The Tacoma narrows bridge motion.

<https://www.youtube.com/watch?v=3mclp9QmCGs>

More on why it collapsed

<https://www.youtube.com/watch?v=mXTSnZgrfxM>



Section 3.6 Forced Oscillations and Resonance

Damped forced oscillators

Let's now put our dash pot back in and study equations like

$$mx'' + cx' + kx = f(t)$$

We again know that we have to solve the complementary equation ..

$$mx'' + cx' + kx = 0$$

For which we saw that we had 3 cases all for which $x(t) \rightarrow 0$ as $t \rightarrow \infty$

Over damped $c^2 > 4mk$ $r_1 = -\frac{c}{2m} + \frac{\sqrt{c^2 - 4mk}}{2m}, r_2 = -\frac{c}{2m} - \frac{\sqrt{c^2 - 4mk}}{2m}$

$$x(t) = c_1 e^{\left(-\frac{c}{2m} + \frac{\sqrt{c^2 - 4mk}}{2m}\right)t} + c_2 e^{\left(-\frac{c}{2m} - \frac{\sqrt{c^2 - 4mk}}{2m}\right)t}$$

Critically damped $c^2 = 4mk, r_1 = \frac{-c}{2m}$

$$x(t) = c_1 e^{\frac{-c}{2m}t} + c_2 t e^{\frac{-c}{2m}t} = e^{\frac{-c}{2m}t} (c_1 + c_2 t)$$

Underdamped case $c^2 < 4km$ $r_1 = -\frac{c}{2m} + \frac{\sqrt{4mk - c^2}}{2m}i, r_2 = -\frac{c}{2m} - \frac{\sqrt{4mk - c^2}}{2m}i$

$$x(t) = e^{-\left(\frac{c}{2m}t\right)} \left[c_1 \cos\left(\frac{\sqrt{4mk - c^2}}{2m}t\right) + c_2 \sin\left(\frac{\sqrt{4mk - c^2}}{2m}t\right) \right]$$

Section 3.6 Forced Oscillations and Resonance

So no matter what if we have damping in the equation $mx'' + cx + kx = f(t)$ the complementary solution $x_c \rightarrow 0$ as $t \rightarrow \infty$. When a solution does this we call it a transient solution. However in our forced case we will also have the particular equation which will dominate as time goes on and we will call it the steady periodic solution. Let's find this for our problem

$$mx'' + cx + kx = F_0 \cos(\omega t)$$

All of the solutions to the homogeneous equation have exponentials on them so we know we have no repeats with the usual guess from the method of undetermined coefficients.

$$x_p = A \cos(\omega t) + B \sin(\omega t)$$

If we plug this in to find our undetermined coefficients after collecting like terms and equating the coefficients on both sides

$$x_p'' = -\omega^2 A \cos(\omega t) + \omega^2 B \sin(\omega t) \quad x_p' = -\omega A \sin(\omega t) + \omega B \cos(\omega t) \quad x_p = A \cos(\omega t) + B \sin(\omega t)$$

$$mx'' + cx + kx = F_0 \cos(\omega t)$$

We end up with the equations:

$$(k - m\omega^2)A + c\omega B = F_0 \quad -c\omega A + (k - m\omega^2)B = 0$$

Which we can solve to find that

$$A = \frac{k - m\omega^2}{(k - m\omega^2)^2 + (c\omega)^2}, \quad B = \frac{c\omega F_0}{(k - m\omega^2)^2 + (c\omega)^2} \sin(\omega t)$$

And so

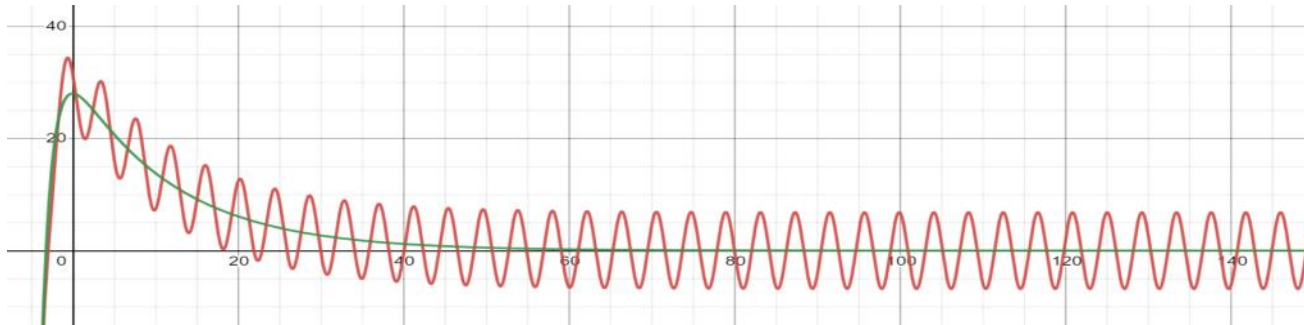
$$x_p(t) = \frac{k - m\omega^2}{(k - m\omega^2)^2 + (c\omega)^2} \cos(\omega t) + \frac{c\omega F_0}{(k - m\omega^2)^2 + (c\omega)^2} \sin(\omega t)$$

Section 3.6 Forced Oscillations and Resonance

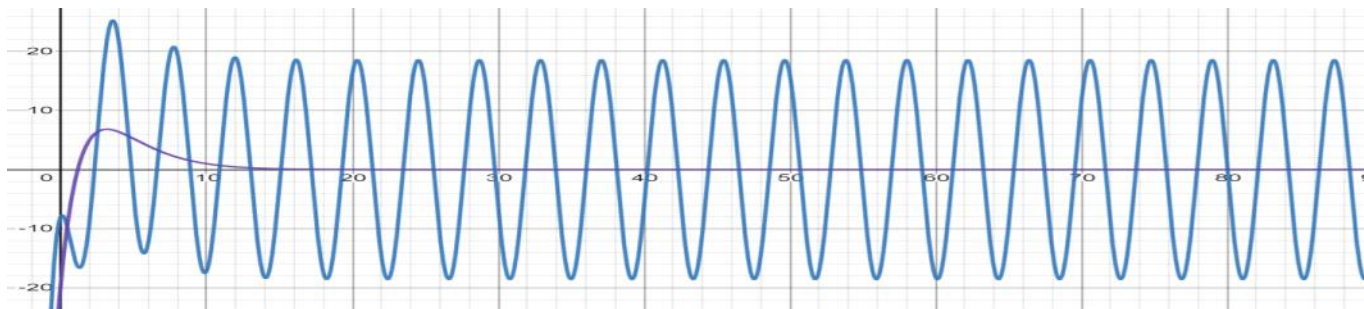
So we end up with
$$x_p(t) = \frac{k - m\omega^2}{(k - m\omega^2)^2 + (c\omega)^2} \cos(\omega t) + \frac{c\omega F_0}{(k - m\omega^2)^2 + (c\omega)^2} \sin(\omega t)$$

So we could have three possibilities depending on c, m, k

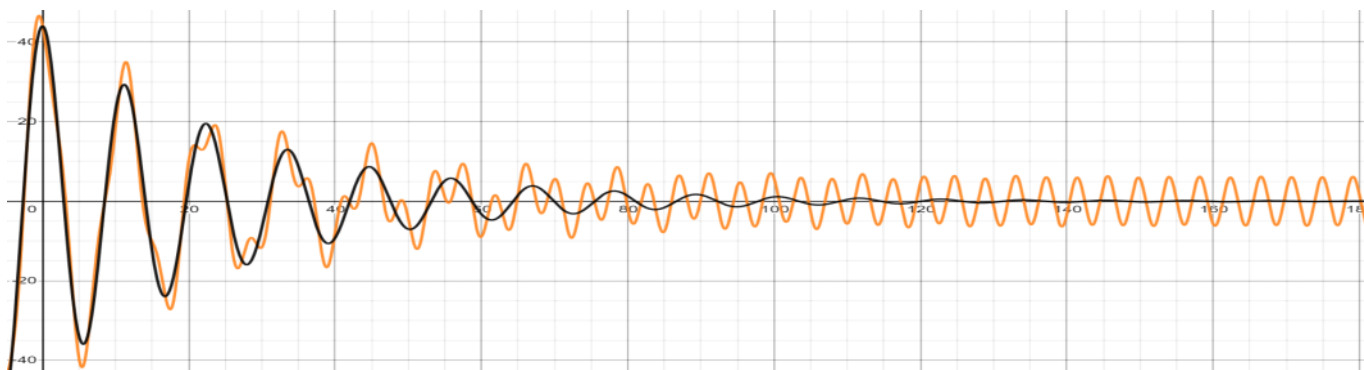
$$x(t) = c_1 e^{\left(-\frac{c}{2m} + \frac{\sqrt{c^2 - 4mk}}{2m}\right)t} + c_2 e^{\left(-\frac{c}{2m} - \frac{\sqrt{c^2 - 4mk}}{2m}\right)t} + \frac{k - m\omega^2}{(k - m\omega^2)^2 + (c\omega)^2} \cos(\omega t) + \frac{c\omega F_0}{(k - m\omega^2)^2 + (c\omega)^2} \sin(\omega t)$$



$$x(t) = c_1 e^{\frac{-c}{2m}t} + c_2 t e^{\frac{-c}{2m}t} = e^{\frac{-c}{2m}t} (c_1 + c_2 t) + \frac{k - m\omega^2}{(k - m\omega^2)^2 + (c\omega)^2} \cos(\omega t) + \frac{c\omega F_0}{(k - m\omega^2)^2 + (c\omega)^2} \sin(\omega t)$$



$$x(t) = e^{-\left(\frac{c}{2m}t\right)} \left[c_1 \cos\left(\frac{\sqrt{4mk - c^2}}{2m}t\right) + c_2 \sin\left(\frac{\sqrt{4mk - c^2}}{2m}t\right) \right] + \frac{k - m\omega^2}{(k - m\omega^2)^2 + (c\omega)^2} \cos(\omega t) + \frac{c\omega F_0}{(k - m\omega^2)^2 + (c\omega)^2} \sin(\omega t)$$



Let's go look at what happens:

<https://www.desmos.com/calculator/mo32shcwyl>

Section 3.6 Forced Oscillations and Resonance

Other forcing functions.... There might be a case where we don't just have a nice sin or cos forcing term . Consider the equation

$$mx'' + kx = \cos^3(\omega t)$$

For your particular solution you could use variation of parameters or in this case a nice trig identity does the trick.

SQUARES $\sin^2 u = \frac{1 - \cos 2u}{2}$ $\cos^2 u = \frac{1 + \cos 2u}{2}$ $\tan^2 u = \frac{1 - \cos 2u}{1 + \cos 2u}$	FOURTHS $\sin^4 u = \frac{3 - 4 \cos 2u + \cos 4u}{8}$ $\cos^4 u = \frac{3 + 4 \cos 2u + \cos 4u}{8}$ $\cos^4 u = \frac{3 - 4 \cos 2u + \cos 4u}{3 + 4 \cos 2u + \cos 4u}$
CUBES $\sin^3 u = \frac{3 \sin u - \sin 3u}{4}$ $\cos^3 u = \frac{3 \cos u + \cos 3u}{4}$ $\tan^3 u = \frac{3 \sin u - \sin 3u}{3 \cos u + \cos 3u}$	FIFTHS $\sin^5 u = \frac{10 \sin u - 5 \sin 3u + \sin 5u}{16}$ $\cos^5 u = \frac{10 \cos u + 5 \cos 3u + \cos 5u}{16}$ $\tan^5 u = \frac{10 \sin u - 5 \sin 3u + \sin 5u}{10 \cos u + 5 \cos 3u + \cos 5u}$

Applying the power reduction formula for $\cos^3(x)$ we get

$$mx'' + kx = \frac{3}{4}\cos(\omega t) + \frac{1}{4}\cos(3\omega t)$$

Now we would just make the guess

$$x_p = A\cos(\omega t) + B\sin(\omega t) + C\cos(3\omega t) + D\sin(3\omega t)$$

Trig identities are your friends!