

Section 4.1 First Order Systems and Applications

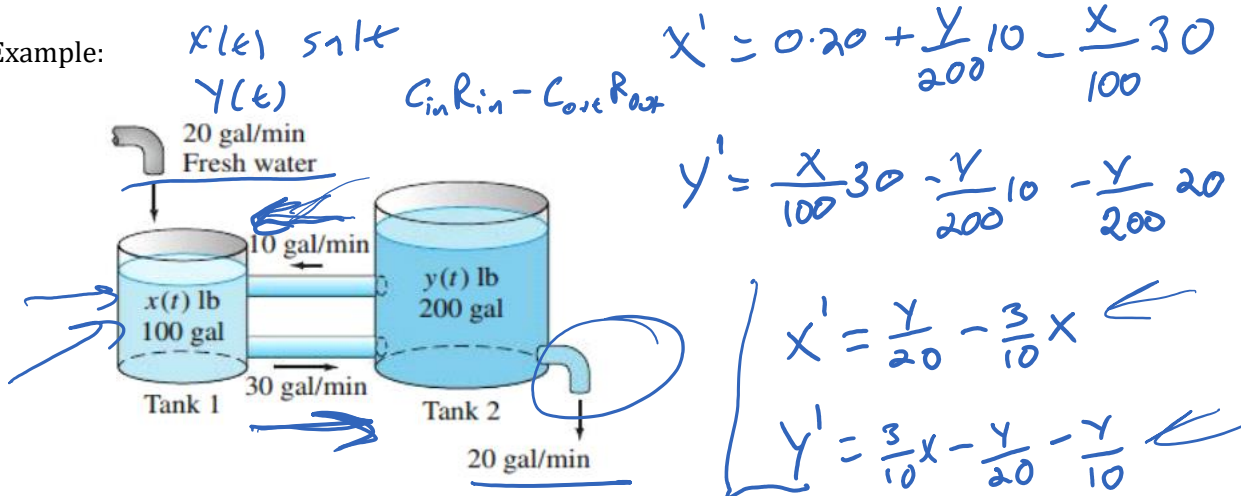
In this section we will be introduced to the concept of a system of differential equations. Systems of differential equations describe how multiple variables evolve together and how they depend on one another.

A first order system of differential equations has a form,

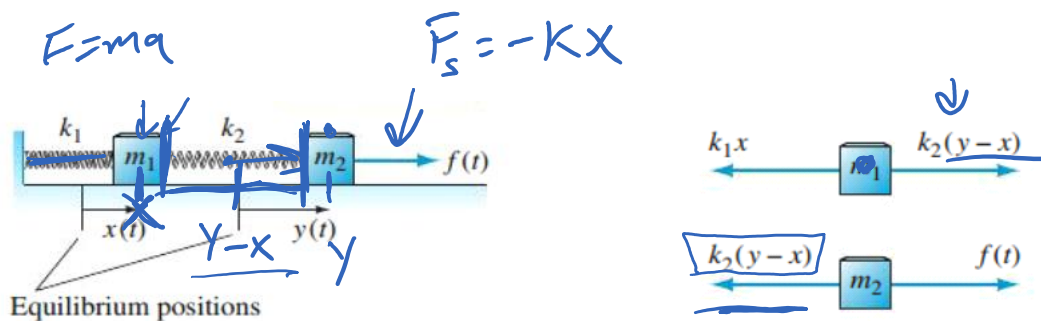
$$\left. \begin{aligned} x_1' + x_2 + x_3' + 4x_4 \\ f_1(t, x_1, x_2, \dots, x_n, x_1', x_2', \dots, x_n') = 0 \\ f_2(t, x_1, x_2, \dots, x_n, x_1', x_2', \dots, x_n') = 0 \\ \vdots \\ f_n(t, x_1, x_2, \dots, x_n, x_1', x_2', \dots, x_n') = 0 \end{aligned} \right\} \begin{array}{l} n \text{ equations} \\ n \text{ variables} \end{array}$$

Note that a first order system only involves up to the first derivative.

Example:



We can also have second order systems and of course more.



$$m_1 x'' = -k_1 x + k_2 (y - x)$$

$$m_2 y'' = -k_2 (y - x) + f(t)$$

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One reason we focus on first order systems is that higher order systems can be transformed into first order systems, which as we will learn have nice solution methods.

Consider the n^{th} order equation $\frac{d^n x}{dt^n} = f(t, x, x', x'', \dots, x^{(n-1)})$

To transform this into a first order system we define the new variables

$$x_1 = x, x_2 = x', x_3 = x'', \dots, x_n = x^{(n-1)}$$

Which gives us the system:

$$\begin{aligned} x_1 &= x \\ x_2 &= x' \\ x_3 &= x'' \\ &\vdots \\ x_n &= x^{(n-1)} \\ x^{(n)} &= f(t, x_1, x_2, x_3, \dots, x^{(n-1)}) \end{aligned}$$

Let's do a more concrete example, we will transform $x^{(3)} + 3x'' + 2x' - 5x = \sin(2t)$ into a system of equations.

$$\begin{aligned} x_1 &= x & x_1' &= x' \\ x_2 &= x' & x_2' &= x'' \\ x_3 &= x'' & x_3' &= x^{(3)} \end{aligned}$$

$$\begin{aligned} x^{(3)} + 3x'' + 2x' - 5x &= \sin(2t) \\ x_3' + 3x_3 + 2x_2 - 5x_1 &= \sin(2t) \\ x_3' &= \sin(2t) - 3x_3 - 2x_2 + 5x_1 \\ x_2' &= x_3 \\ x_1' &= x_2 \end{aligned}$$

It is worth noting that we could have just solved the original equation above with what we have learned so far, but not every equation is linear like that above. Nonlinear equations like,

$$x'' = x^3 + (x')^3$$

Is difficult to solve but as a system of equations there are numerical methods that apply.

$$\begin{aligned} x_1 &= x & x_1' &= x_2 \\ x_2 &= x' & x_2' &= x_1^3 + x_2^3 \end{aligned}$$

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Let's transform one more for practice.

$$\begin{aligned} x_1 &= x \\ x_2 &= x' \\ x_2' &= x'' \\ y_1 &= y \\ y_2 &= y' \\ y_2' &= y'' \end{aligned}$$

$$\begin{aligned} x'' &= -6x + 2y \\ y'' &= 2x - 2y + 40 \sin(3t) \end{aligned}$$

$$x'' = 6x + 2y$$

$$x_2' = 6x_1 + 2y_1$$

$$y'' = 2x - 2y + 40 \sin(3t)$$

$$y_2' = 2x_1 - 2y_1 + 40 \sin(3t)$$

$$\begin{aligned} x_1' &= x_2 \\ y_1' &= y_2 \end{aligned}$$

$$\begin{aligned} \text{of. } x_1 &= x & x_3 &= y \\ x_2 &= x' & x_4 &= y' \\ x_2' &= x'' & x_4' &= y'' \\ x_1' &= x_2 & x_3' &= x_4 \\ x_2' &= 6x_1 + 2x_3 & x_4' &= 2x_1 - 2x_3 + 40 \sin(3t) \end{aligned}$$

Let's look at an example of a system and some tools that help us analyze them.

Suppose we have a particle floating on the surface of a pool with a position $[x(t), y(t)] = xi + yi$. Suppose that we know the velocity of the particle is $[x'(t), y'(t)] = [-2y, \frac{1}{2}x]$ that is the x component of the velocity in two dimensions is $x' = -2y$ and the y component is $y' = \frac{1}{2}x$. Giving us a system of differential equations

Just like we used slope fields in previous sections to get a sense of what solutions to a given differential equation looks like we can look at the *direction field* when dealing with first order systems.

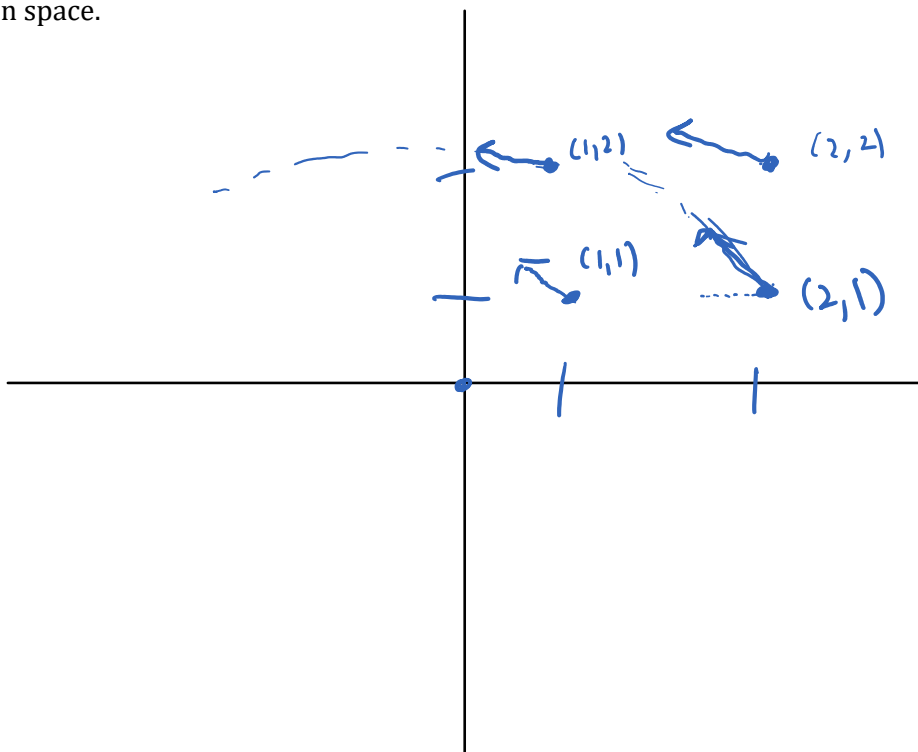
The direction field of a first order system is a plot where we place a velocity vector defined by the system of equations at each point in space.

$$\begin{aligned} x' &= -2y \\ y' &= \frac{1}{2}x \end{aligned}$$

at $(1, 1)$

$$\begin{aligned} [x', y'] &= [-2(1), \frac{1}{2}(1)] \\ &= [-2, \frac{1}{2}] \end{aligned}$$

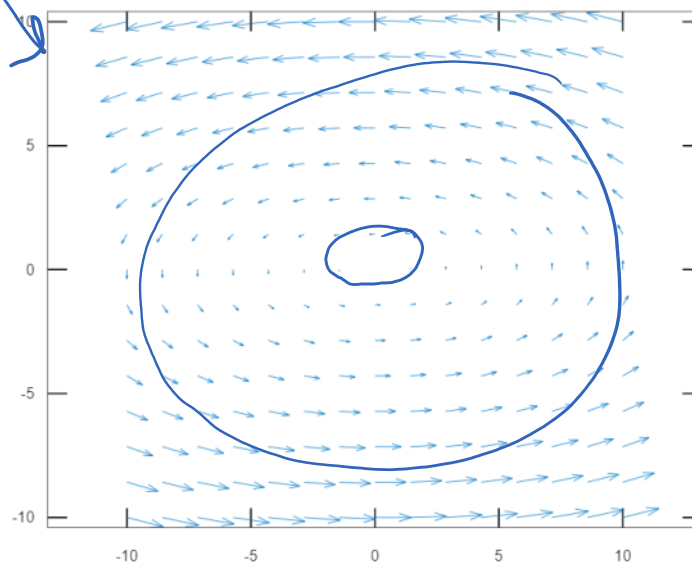
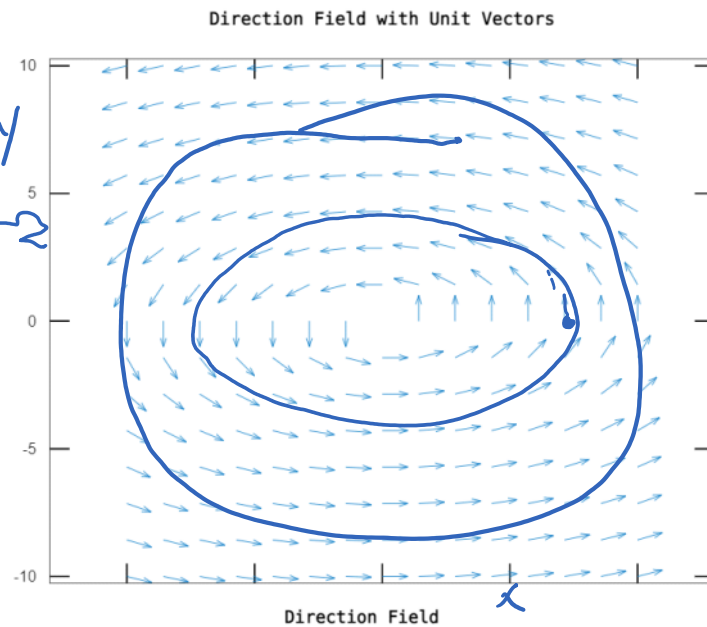
$$\begin{aligned} \text{at } (1, 2) \\ [x', y'] &= [-2(2), \frac{1}{2}(1)] \\ &= [-4, \frac{1}{2}] \end{aligned}$$



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Of course just as before we can use a computer to help us make these plots quickly and easily.

```
1 x=linspace(-10,10,15);% define the x range in space
2 y=linspace(-10,10,15);% define the y range in space
3 [X,Y]=meshgrid(x,y); %make a grid of points with the given x and y ranges
4 %let u=x'
5 u=-2*Y;
6 %let v= y'
7 v=(1/2)*X;
8 %quiver(X,Y,u,v,0.8) makes the direction field plot scaling the vectors by the scale factor
9 quiver(X,Y,u,v,0.8)
10 title('Direction Field')
11 axis equal
12 %if we want unit vectors we can scale the components by the vector magnitudes and set scaling to zero to turn it off.
13 figure % prevents new plot from going over previous one.
14 quiver(X,Y,u./sqrt(u.^2+v.^2),v./sqrt(u.^2+v.^2),0)
15 title('Direction Field with Unit Vectors')
16 axis equal
```



You can get the code for directions fields here:

<https://octav.onl/okhes701>

From <<https://octave-online.net/bucket~HWSGzvpowZPrE2xLtY6Drp>>

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Let's take a look at this system and solve it (the old way this time but we will learn how to solve systems soon)

$$\begin{aligned} x'' &= -2y' \\ x' &= -2y \\ y' &= \frac{1}{2}x \end{aligned} \quad \left. \begin{array}{l} x' = -2y \\ y' = \frac{1}{2}x \end{array} \right\} \quad x'' = -2\left(\frac{1}{2}x\right) \quad x'' = -x \quad \underline{x'' + x = 0}$$

Note that $x'' = -2y'$ so then $x'' = -2\left(\frac{1}{2}x\right)$ or $x'' + x = 0$

What is our solution for x ?

$$x'' + x = 0 \quad r^2 + 1 = 0 \quad r^2 = -1 \quad r = \pm i$$

$$x = A \cos(t) + B \sin(t)$$

Let's suppose $x(0) = 5$, $x'(0) = 0$

$$\begin{aligned} x(0) &= 5 = A \cos(0) + B \sin(0) \quad A = 5 \\ x'(0) &= 0 = -A \sin(0) + B \cos(0) \rightarrow x'(0) = 0 = -A \sin(0) + B \cos(0) \\ B &= 0 \end{aligned}$$

How can we get y ?

$$y(0) = 0$$

$$x = 5 \cos(t)$$

$$y' = \frac{1}{2} 5 \cos(t) \Rightarrow y = \frac{5}{2} \sin(t) + C$$

$$y(0) = 0 + C = 0 \quad C = 0$$

What does this solution tell us?

$$[x(t), y(t)]$$

$$= [5 \cos(t), \frac{5}{2} \sin(t)]$$

$$x^2 + y^2 =$$

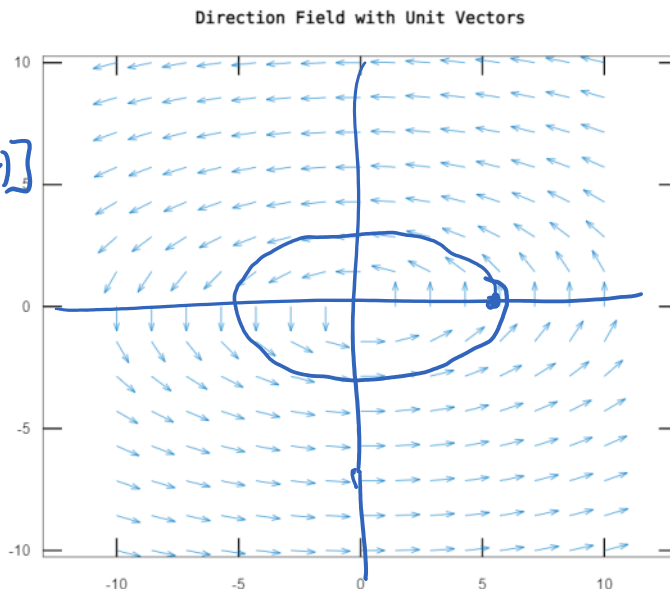
$$25 \cos^2(t) + \frac{25}{4} \sin^2(t)$$

$$\left(\frac{x}{5}\right)^2 + y^2 =$$

$$\frac{25}{4} \cos^2 t + \frac{25}{4} \sin^2(t)$$

$$= \frac{25}{4} (\cos^2(t) + \sin^2(t)) \rightarrow \frac{x^2}{4} + y^2 = \frac{25}{4}$$

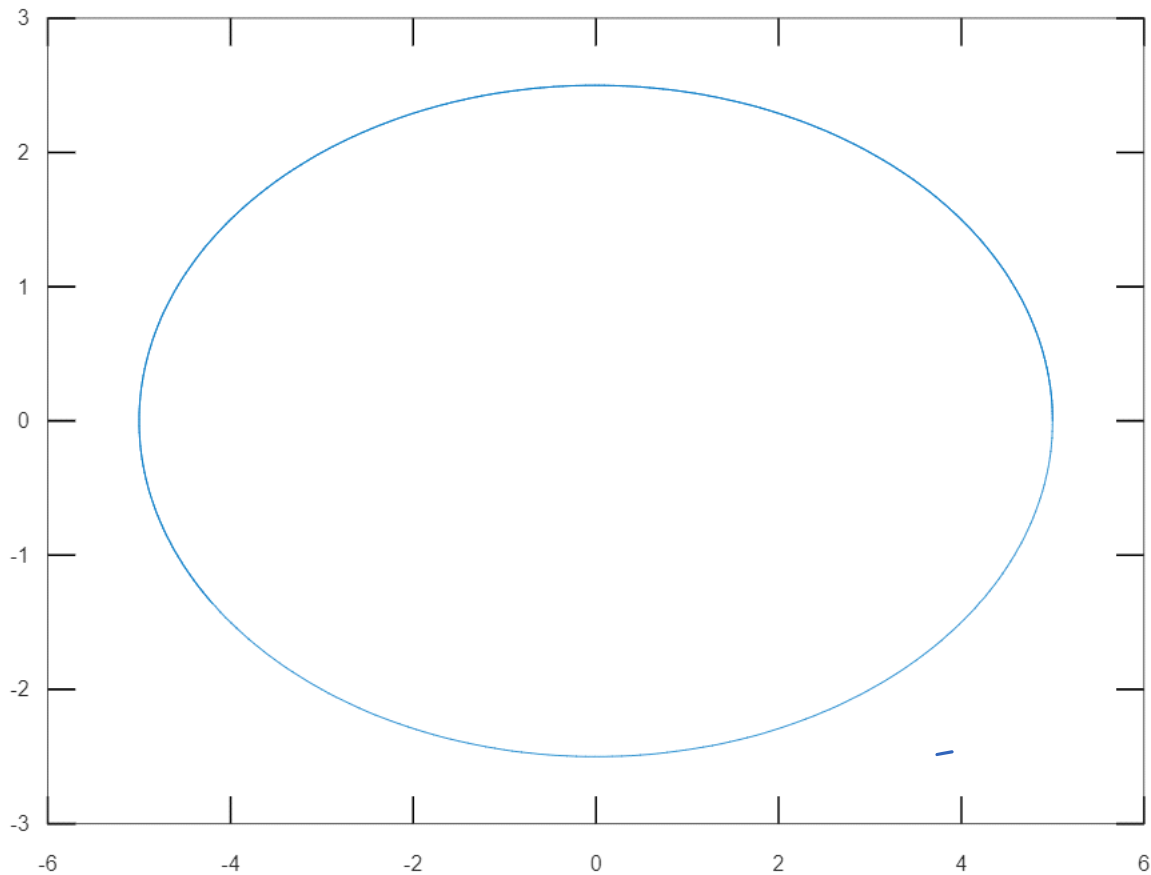
We can also just plot the parametric equations in the plane...



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$$x = 5 \cos(t), y = \frac{5}{2} \sin(t)$$

```
1 t=linspace(0,10,500)
2 x=5*cos(t);
3 y=(1/2)*5*sin(t);
4 plot(x,y)
```



We can also make a movie if we like!

```
1 for t=[0:0.1:20]
2 x=5*cos(t);
3 y=(1/2)*5*sin(t);
4 plot(x,y,'o')
5 xlim([-6 6])
6 ylim([-6 6])
7 pause(0.1)
8 end
```

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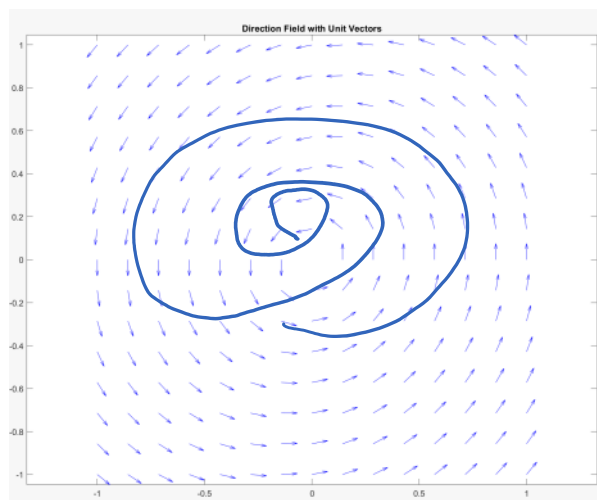
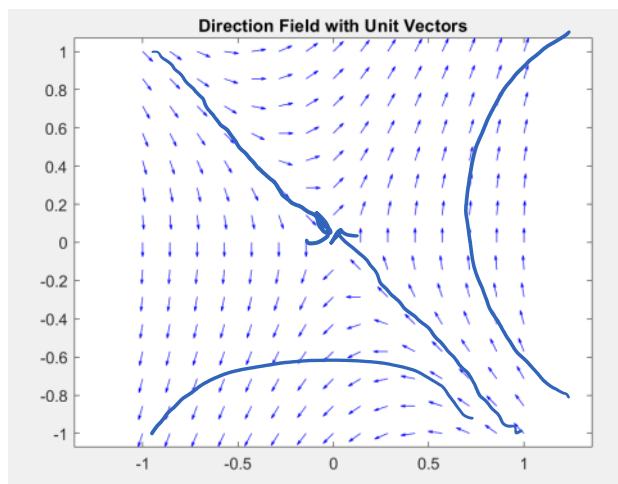
Let's look at two other linear systems

$$\begin{aligned} x' &= y \\ y' &= 2x + y \\ x'' &= y' \\ x'' &= 2x + x' \end{aligned}$$

and

$$\begin{aligned} x' &= -y \\ y' &= (1.01)x - 0.2y \\ x'' &= y' \\ x'' &= -1.01x + 0.2y \\ x'' &= -1.01x - 0.2x' \end{aligned}$$

What do the direction fields look like?



What are their solutions?

$$x'' - x' - 2x = 0$$

$$r^2 - r - 2 = 0$$

$$(r+1)(r-2)$$

$$\begin{aligned} x &= c_1 e^{-t} + c_2 e^{2t} \\ y &= -c_1 e^{-t} + 2c_2 e^{2t} \end{aligned}$$

$$\Rightarrow x'' + 0.2x' + 1.01x = 0$$

$$r = \frac{-0.2 \pm \sqrt{(0.2)^2 - 4(1.01)}}{2}$$

complex roots

$$r = \frac{-0.2 \pm \sqrt{1.01 - 4(1.01)}}{2}$$

$$r_1 = -a + bi \quad r_2 = -a - bi$$

$$x = e^{-at} (A \cos(bt) + B \sin(bt))$$

$$\begin{aligned} y &= -x' = -\left[e^{-at} (-A \sin(bt) + B \cos(bt)) \right] + e^{-at} (-bA \sin(bt) - bB \cos(bt)) \\ x &= e^{-at} (\text{trig}) \quad y = e^{-at} (\text{trig}) \end{aligned}$$

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Let's make some plots and movies for

$$\begin{aligned}x' &= -y \\ y' &= (1.01)x - 0.2y\end{aligned}$$

Our general solution is $e^{-\frac{t}{10}}(A\cos(t) + B\sin(t))$ suppose we have initial data $x(0) = 0, y(0) = -1$

Let's make some plots and movies!

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Some terminology and a theorem

For a first order linear system of the form:

$$\begin{aligned}x_1' &= p_{11}(t)x_1 + p_{12}(t)x_2 + \cdots + p_{1n}x_n + f_1(t) \\x_2' &= p_{21}(t)x_1 + p_{22}(t)x_2 + \cdots + p_{2n}x_n + f_2(t) \\&\vdots \\x_n' &= p_{n1}(t)x_1 + p_{n2}(t)x_2 + \cdots + p_{nn}x_n + f_n(t)\end{aligned}$$

We say it is **homogeneous** if the functions $f_1, f_2, \dots, f_n$ are all zero and that it is **nonhomogeneous** if any of them are not zero.

Existence and Uniqueness for Linear Systems

Suppose that the functions $p_{11}, p_{12}, \dots, p_{nn}$ and the functions $f_1, f_2, \dots, f_n$ are continuous on the open interval I containing the point a . Then, given the n numbers $b_1, b_2, \dots, b_n$ the system

$$\begin{aligned}x_1' &= p_{11}(t)x_1 + p_{12}(t)x_2 + \cdots + p_{1n}x_n + f_1(t) \\x_2' &= p_{21}(t)x_1 + p_{22}(t)x_2 + \cdots + p_{2n}x_n + f_2(t) \\&\vdots \\x_n' &= p_{n1}(t)x_1 + p_{n2}(t)x_2 + \cdots + p_{nn}x_n + f_n(t)\end{aligned}$$

has a unique solution on the entire interval I that satisfies the n initial conditions

$$x_1(a) = b_1, x_2(a) = b_2, \dots, x_n(a) = b_n$$