

Section 5.1 Matrices and Linear Systems

Matrices are mathematical objects and we can do fun things like add them and multiply them. The set of $n \times n$ matrices itself makes up something called a non-commutative Ring, not a topic for this class but worth a google if you are interested. A Ring, in short, is an algebraic structure for which its members satisfy specific properties.

So we know how to add vectors but how we add matrices? If $A, B \in R^{n \times m}$, the set of $n \times m$ matrices.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix}$$

Handwritten notes: "rows" with an arrow pointing to the rows of matrix A, and "columns" with an arrow pointing to the columns of matrix A.

Then

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1m} + b_{1m} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2m} + b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} + b_{n1} & a_{n2} + b_{n2} & \cdots & a_{nm} + b_{nm} \end{bmatrix}$$

That is we add them component wise.

Example: Calculate

$$A + B = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 2 & 7 & 4 \\ 0 & 8 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 4 & 3 \\ 6 & 1 & 0 \\ 2 & 5 & 2 \\ 9 & 1 & 10 \end{bmatrix}$$

Handwritten blue circles around corresponding elements in the two matrices, with arrows indicating the addition process.

$$\begin{bmatrix} 1+2 & 2+4 & 3+3 \\ -1+6 & 0+1 & 2+0 \\ 2+2 & 7+5 & 4+2 \\ 0+9 & 8+1 & 1+10 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 6 \\ 5 & 1 & 2 \\ 4 & 12 & 6 \\ 9 & 9 & 11 \end{bmatrix}$$

What about $B + A$?

$$A + B = B + A$$

Section 5.1 Matrices and Linear Systems

Scalar Multiplication

Let $c \in R$ and $A \in R^{n \times m}$ be a scalar then

$$cA = cA = c \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} = \begin{bmatrix} ca_{11} & ca_{12} & \cdots & ca_{1m} \\ ca_{21} & ca_{22} & \cdots & ca_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{n1} & ca_{n2} & \cdots & ca_{nm} \end{bmatrix}$$

Example: Calculate

$$4 \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 2 & 7 & 4 \\ 0 & 8 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 12 \\ -4 & 0 & 8 \\ 8 & 28 & 16 \\ 0 & 32 & 4 \end{bmatrix}$$

Theorem Let s and t be scalars and $A, B, C \in R^{n \times m}$ and 0_{nm} a matrix of zeros of size $n \times m$ then,

- (a) $A + B = B + A$
- (b) $s(A + B) = sA + sB$
- (c) $(s + t)A = sA + tA$
- (d) $(A + B) + C = A + (B + C)$
- (e) $(st)A = s(tA)$
- (f) $(A + 0_{nm}) = A$

Section 5.1 Matrices and Linear Systems

Matrix Multiplication

Let $A = [\vec{a}_1 \ \vec{a}_2 \ \dots \ \vec{a}_k] \in R^{n \times k}$ where $\vec{a}_i \in R^n$ and $B = [\vec{b}_1 \ \vec{b}_2 \ \dots \ \vec{b}_m] \in R^{k \times m}$ and $\vec{b}_i \in R^k$ then the matrix product is defined by:

$$AB = [A\vec{b}_1 \ A\vec{b}_2 \ \dots \ A\vec{b}_m]$$

$$= [(b_{11}\vec{a}_1 + b_{21}\vec{a}_2 + \dots + b_{k1}\vec{a}_k) \ (b_{12}\vec{a}_1 + b_{22}\vec{a}_2 + \dots + b_{k2}\vec{a}_k) + \dots + (b_{1m}\vec{a}_1 + b_{2m}\vec{a}_2 + \dots + b_{km}\vec{a}_k)]$$

$$= \left(\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} b_{11} + \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} b_{21} + \dots + \begin{bmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{nk} \end{bmatrix} b_{k1} \right) + \left(\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} b_{12} + \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} b_{22} + \dots + \begin{bmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{nk} \end{bmatrix} b_{k2} \right) + \dots + \left(\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} b_{1m} + \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} b_{2m} + \dots + \begin{bmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{nk} \end{bmatrix} b_{km} \right)$$

$$= \begin{bmatrix} b_{11}a_{11} + b_{21}a_{12} + \dots + b_{k1}a_{1k} & b_{12}a_{11} + b_{22}a_{12} + \dots + b_{k2}a_{1k} & \dots & b_{1m}a_{11} + b_{2m}a_{12} + \dots + b_{km}a_{1k} \\ b_{11}a_{21} + b_{21}a_{22} + \dots + b_{k1}a_{2k} & b_{12}a_{21} + b_{22}a_{22} + \dots + b_{k2}a_{2k} & \dots & b_{1m}a_{21} + b_{2m}a_{22} + \dots + b_{km}a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ b_{11}a_{n1} + b_{21}a_{n2} + \dots + b_{k1}a_{nk} & b_{12}a_{n1} + b_{22}a_{n2} + \dots + b_{k2}a_{nk} & \dots & b_{1m}a_{n1} + b_{2m}a_{n2} + \dots + b_{km}a_{nk} \end{bmatrix}$$

We can also state this in the following way if $AB = C$ then

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \dots + a_{ik}b_{kj}$$

Which is an $n \times m$ matrix

$n \times m$
rows cols

Example: Calculate

$$AB = \begin{bmatrix} 1 & 0 \\ -2 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ 4 & 2 & -1 & 0 \end{bmatrix}$$

$3 \times 2 \quad 2 \times 4 = 3 \times 4$

$BA \quad 2 \times 4 \quad 3 \times 2$
 $n \times m \quad m \times k \quad n \times k$
 $A \quad B = C$

$$\begin{bmatrix} 1(2) + 0(4) & 1(-2) + 0(2) & 1(1) + 0(-1) & 1(1) + 0(0) \\ -1(2) + 2(4) & (-1)(-2) + 2(2) & -1(1) + 2(-1) & -1(1) + 2(0) \\ 3(2) + 1(4) & 3(-2) + 1(2) & 3(1) + 1(-1) & 3(1) + 1(0) \end{bmatrix} = \begin{bmatrix} 2 & -2 & 1 & 1 \\ 6 & 6 & -3 & -1 \\ 10 & -4 & 2 & 3 \end{bmatrix}$$

One way to think about it is that the entry in the i^{th} row and j^{th} column will come from the i^{th} row of the left matrix and the j^{th} column of the right matrix.

What about BA ?

can't do it $2 \times 4 \quad 3 \times 2$ not same $\begin{bmatrix} 2 & -2 & 1 & 1 \\ 4 & 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 3 & 1 \end{bmatrix}$

Section 5.1 Matrices and Linear Systems

In General $AB \neq BA$ for matrices!

Matrices of different sizes

```
octave:2> A=[1,2,3,4;5,6,7,8]
```

A =

1	2	3	4
5	6	7	8

```
octave:3> B=[1 2; 3 4; 5 6; 7 8]
```

B =

1	2
3	4
5	6
7	8

```
octave:4> A*B
```

ans =

50	60
114	140

```
octave:5> B*A
```

ans =

11	14	17	20
23	30	37	44
35	46	57	68
47	62	77	92

Matrices of the same size

```
octave:6> A=[1,2,3;4,5,6;7,8,9]
```

A =

1	2	3
4	5	6
7	8	9

```
octave:7> B=[9,8,7;6,5,4;3,2,1]
```

B =

9	8	7
6	5	4
3	2	1

```
octave:8> A*B
```

ans =

30	24	18
84	69	54
138	114	90

```
octave:9> B*A
```

ans =

90	114	138
54	69	84
18	24	30

```
octave:10> A.*B
```

ans =

9	16	21
24	25	24
21	16	9

```
octave:11> B.*A
```

ans =

9	16	21
24	25	24
21	16	9

Notice here they were the same, but this was **not** matrix multiplication. In Matlab or Octave * and .* (dot times) are very different. The .* operator multiplies component wise $a_{11}b_{11}$ $a_{12}b_{12}$ etc. In fact it is the Hadamard product we mentioned when first talking about vectors. If the matrices are of different sizes you cannot do the Hadamard product.

Section 5.1 Matrices and Linear Systems

We stated in Theorem 3.11 that $A + 0_{nm} = A$ the zero matrix is often referred to as the additive identity for matrix addition, one might wonder then, is there a multiplicative identity for matrix multiplication?

The identity matrix: Let $A \in R^{n \times m}$ then the identity matrices will be $I_m \in R^{m \times m}$, $I_n \in R^{n \times n}$ and will have ones on the diagonal and zeros everywhere else. That is:

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \quad \text{Further } \underbrace{A I_m}_{n \times n \times n \times m} = \underbrace{I_n A}_{n \times n \times n \times m} = A$$

If $A \in R^{n \times n}$, is a square matrix $n = m$, then $I = I_n = I_m \in R^{n \times n}$ and we can say $AI = IA = A$

Example:

Multiply:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1(1) + 0(3) & 1(2) + 0(4) \\ 0(1) + 3(1) & 0(2) + 3(4) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad m \times n \quad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad n \times m$$

$I \quad A$

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

Section 5.1 Matrices and Linear Systems

Theorem Let s be a scalar and A, B, C be matrices of appropriate sizes. Then

- (a) $A(BC) = (AB)C$
- (b) $A(B + C) = AB + AC$
- (c) $(A + B)C = AC + BC$
- (d) $s(AB) = (sA)B = A(sB)$
- (e) $AI = A$
- (f) $IA = A$

Theorem Let A, B, C be nonzero matrices then,

- (a) In general $AB \neq BA$
- (b) $AB = 0$ **does not** imply that $A = 0$ or $B = 0$.
- (c) $AC = BC$ **does not** imply that $A = B$ or $C = 0$

Example

$$\begin{matrix} 2 \times 2 & 2 \times 2 \\ \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} & \begin{bmatrix} -4 & 2 \\ 2 & -3 \end{bmatrix} \end{matrix}$$

$$\begin{bmatrix} -4 & 2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 2 & -4 \\ -7 & -14 \end{bmatrix}$$

$$\begin{bmatrix} 1(-4) + 2(2) & 1(2) + 2(-3) \\ -4 + 12 & 6 - 18 \end{bmatrix} = \begin{bmatrix} 0 & -4 \\ 8 & -12 \end{bmatrix}$$

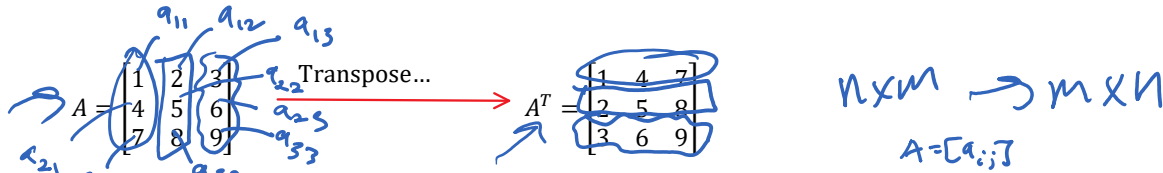
Example Let $A = \begin{bmatrix} -3 & 3 \\ 11 & -3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ Find AB and BC

Section 5.1 Matrices and Linear Systems

The Transpose of a Matrix

The transpose of a matrix is easy, columns become rows and rows become rows become columns

The first column becomes the first row, the second column becomes the second row, the third column becomes the third row and so on.



Notice the diagonal stayed the same! Infact we are just moving a_{ij} to a_{ji} so on the diagonal $a_{ii} = a_{ii}$

Theorem 3.15: Let $A, B \in \mathbb{R}^{n \times m}$, $C \in \mathbb{R}^{m \times k}$ and s scalar. Then

- (a) $(A + B)^T = A^T + B^T$
- (b) $(sA)^T = sA^T$
- (c) $(AC)^T = C^T A^T$

Short Proof of (a) good enough for quiz credit as well

Let $A+B=C$ with $A=[a_{ij}]$, $B=[b_{ij}]$, $C=[c_{ij}]=[a_{ij}+b_{ij}]$
 Then let $A^T=[a_{ji}^T]$, $B^T=[b_{ji}^T]$, $C^T=[c_{ji}^T]$, by the definition
 of transposition, $a_{ji}^T=a_{ij}$, $b_{ji}^T=b_{ij}$, $c_{ji}^T=c_{ij}=a_{ij}+b_{ij}$.
 Since $A^T+B^T=[a_{ji}^T+b_{ji}^T]=[a_{ij}+b_{ij}]=[c_{ij}]=[c_{ji}^T]$
 and $[c_{ji}^T]=[a_{ij}+b_{ij}]^T=(A+B)^T \Rightarrow A^T+B^T=(A+B)^T$

Quiz problem: Prove (a)

We prove this by (1) calculating $A^T + B^T$ and (2) calculating $(A + B)^T$ and showing we get the same thing! Note tht we are fixing the values of the components in A and B and just moving them, so after the transposition the subscripts no longer correspond to the position in the matrix, rather the value itself. So if a_{32} was in row 3 col 2 in A and a_{23} in row 2 col 3 in A^T then $a_{23}^T = a_{32}$ which we put in the spot of a_{23}^T . I think the colors make things clear.

①

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix} \xrightarrow{\text{add}} A+B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & a_{mn}+b_{mn} \end{bmatrix}$$

↓ columns to rows

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix}, B^T = \begin{bmatrix} b_{11} & b_{21} & \dots & b_{m1} \\ b_{12} & b_{22} & \dots & b_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ b_{1n} & b_{2n} & \dots & b_{mn} \end{bmatrix} \xrightarrow{\text{add}} A^T+B^T = \begin{bmatrix} a_{11}+b_{11} & a_{21}+b_{21} & \dots & a_{m1}+b_{m1} \\ a_{12}+b_{12} & a_{22}+b_{22} & \dots & a_{m2}+b_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n}+b_{1n} & a_{2n}+b_{2n} & \dots & a_{mn}+b_{mn} \end{bmatrix}$$

// Same!

②

$$A+B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & a_{mn}+b_{mn} \end{bmatrix} \xrightarrow{\text{columns to rows}} (A+B)^T = \begin{bmatrix} a_{11}+b_{11} & a_{21}+b_{21} & \dots & a_{m1}+b_{m1} \\ a_{12}+b_{12} & a_{22}+b_{22} & \dots & a_{m2}+b_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n}+b_{1n} & a_{2n}+b_{2n} & \dots & a_{mn}+b_{mn} \end{bmatrix}$$

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Section 5.1 Matrices and Linear equations

The determinant of a matrix is a way to map all of the entries of a matrix to a single number. This single number can be used to determine whether or not a square matrix is invertible or to give us a sense of how much error we might see in a numerical calculation involving matrices.

We build up the definition of the determinant by first defining the easiest ones to calculate.

If $A = [a_1] \in R$ is a 1×1 matrix then the $\det(A) = |A| = a_1$

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a 2×2 matrix then $\det(A) = |A| = ad - bc$

If you recall the inverse of a 2×2 matrix was

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

and it was exactly when $(ad - bc) = 0$ that the matrix was not invertible. Infact:

An $n \times n$ matrix A is invertible if and only if $\det(A) \neq 0$

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then the determinant is given by,

$$\det(A) = |A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

Formulas for determinants of larger matrices can be written down, but they get increasingly messy. On the next page we will learn a procedural way to calculate a determinant which relies on the definitions for smaller matrices here.

Is the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{bmatrix}$ invertible?

$$1(5)(0) + 2(6)(0) + 3(4)(0) - 1(6)(0) + 0 + 0 = 0$$

Section 5.1 The Determinant Function

Let A be the $n \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix}$$

Then the determinant of A is

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

Cofactor Definition for Reference

Let A be an $n \times n$ matrix the cofactor C_{ij} of if the ij^{th} entry a_{ij} is

$$C_{ij} = (-1)^{i+j} \det(M_{ij})$$

where M_{ij} is the submatrix formed by omitting the entries in the i^{th} row and the j^{th} column of A

Ex Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 6 \\ 7 & -1 & 1 \end{bmatrix}$
Find $\det(A)$

Handwritten notes: $a_{12} \quad 1+2=3$
 $+ \quad - \quad +$
 $1+3=4$

$$1 \begin{vmatrix} 1 & 6 \\ -1 & 1 \end{vmatrix} - 0 \begin{vmatrix} 2 & 6 \\ 7 & 1 \end{vmatrix} + 2 \begin{vmatrix} 2 & 1 \\ 7 & -1 \end{vmatrix}$$

$$1(1(-1) - (-1)(6)) - 0 + 2(2(-1) - 7(1))$$

$$7 - 0 + -18 = -11$$

In the calculation above it was nice that we had a 0 in the first row as it reduced the number of calculations.

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 0 & 1 & 6 \\ 0 & -1 & 1 \end{bmatrix} \quad (1) \begin{vmatrix} 1 & 6 \\ -1 & 1 \end{vmatrix} - 0 + 0 = 7$$

There are a lot of zeros in that first column! Using the definition above we would have expanded along the first row, however we can actually expand across any row or column.

Let A be an $n \times n$ matrix then

(a) $\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$ (Expansion across row i) $\leftarrow$

(b) $\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$ (Expansion across row j) $\leftarrow$

(a) and (b) are called the cofactor expansions

Section 5.1 Matrices and Linear Systems

Powers of a matrix

Just like we would say x multiplied by itself n times is written x^n we can do the same with a matrix

$$A^n = \underbrace{A A \cdots A}_{n \text{ times}}$$

Diagonal matrices are particularly nice for this operation.

$$A^n = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix} \cdots \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} a_{11}^n & 0 & 0 & \cdots & 0 \\ 0 & a_{22}^n & 0 & \cdots & 0 \\ 0 & 0 & a_{33}^n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn}^n \end{bmatrix}$$

Ex: Calculate A^3 if $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$

$$A^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 27 \end{bmatrix}$$

Another important form of matrices are Upper Triangular and Lower Triangular matrices.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

Upper Triangular

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

Lower Triangular

Note that an Upper triangular matrix corresponds to echelon form and a diagonal matrix is almost in reduced row echelon form. We can also make the following statements.

- (a) Upper and Lower Triangular matrices are called triangular
- (b) A Diagonal matrix is both Upper Triangular and Lower Triangular
- (c) If A and B are both either Upper Triangular or Lower Triangular so is $A + B$
- (d) Powers of triangular matrices are also triangular

Section 5.1 Matrices and Linear Systems

We define the inverse matrix an $n \times n$ matrix A to be the matrix A^{-1} such that

$$A^{-1}A = I = AA^{-1}$$

$$\frac{1}{2}2 = 1 = 2\frac{1}{2}$$

$$2^{-1} = \frac{1}{2}$$

Example: Let $A = \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix}$ Show $B = A^{-1}$

$$AB = I \quad \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Use the fact that $B = A^{-1}$ to solve the system: $x_1 + 3x_2 = 1$ and $2x_1 + 5x_2 = 2$

$$x_1 + 3x_2 = 1$$

$$2x_1 + 5x_2 = 2$$

$$x_1 + 3x_2 = 1$$

$$2x_1 + 5x_2 = 2$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$A \vec{x} = \vec{b}$$

$$A^{-1}A\vec{x} = A^{-1}\vec{b}$$

$$2 \times 2 \quad 2 \times 1 \quad 2 \times 1 \quad I \quad \vec{x} = A^{-1}\vec{b}$$

$$\begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -5 & 3 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$I \quad \vec{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -5(1) + 3(2) \\ 2(1) - 1(2) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Section 5.1 Matrices and Linear Systems

Some nice formulas for nice matrices.

If a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible then $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$$A^{-1}A = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

From this formula, we can tell A would not have an inverse if $ad - bc = 0$, we can't divide by zero!

If a diagonal matrix $A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$ is invertible then $A^{-1} = \begin{bmatrix} 1/a_{11} & 0 & 0 & \cdots & 0 \\ 0 & 1/a_{22} & 0 & \cdots & 0 \\ 0 & 0 & 1/a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1/a_{nn} \end{bmatrix}$

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} 1/a_{11} & 0 & 0 & \cdots & 0 \\ 0 & 1/a_{22} & 0 & \cdots & 0 \\ 0 & 0 & 1/a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1/a_{nn} \end{bmatrix} = \begin{bmatrix} a_{11}/a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22}/a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33}/a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn}/a_{nn} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$(\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix})^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$

Here we could see that if any of the a'_{ii} 's are zero the diagonal matrix would not have an inverse.

Section 5.1 Matrices and Linear Systems

Matrix valued functions.

We can also put functions into our matrices to get matrix valued functions!

$$A(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1m}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2m}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nm}(t) \end{bmatrix}$$

We can take derivatives too, if each function in the matrix is differentiable that is. The derivative of our matrix valued function is defined as,

$$\frac{dA(t)}{dt} = \begin{bmatrix} a'_{11}(t) & a'_{12}(t) & \cdots & a'_{1m}(t) \\ a'_{21}(t) & a'_{22}(t) & \cdots & a'_{2m}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a'_{n1}(t) & a'_{n2}(t) & \cdots & a'_{nm}(t) \end{bmatrix} = \left[\frac{da_{ij}}{dt} \right]$$

That is we just take the derivatives of each element of the matrix.

Example: Find the derivatives of $x(t) = \begin{bmatrix} e^t \\ t^2 \\ \cos(t) \end{bmatrix}$ and $A(t) = \begin{bmatrix} \cos(t) & \sin(t) \\ 2 & \ln(t) \end{bmatrix}$

$$x'(t) = \begin{bmatrix} e^t \\ 2t \\ -\sin(t) \end{bmatrix}$$

$$A'(t) = \begin{bmatrix} -\sin(t) & \cos(t) \\ 0 & 1/t \end{bmatrix}$$

Section 5.1 Matrices and Linear Systems

We also get to keep some familiar properties of differentiation when it comes to matrices.

Sum rule

$$\frac{d}{dt}(A + B) = \frac{dA}{dt} + \frac{dB}{dt}$$

Product rule

$$\frac{d}{dt}(AB) = \frac{dA}{dt}B + A\frac{dB}{dt}$$

Constant matrices

$$\frac{d}{dt}(CA) = C\frac{dA}{dt} \quad \frac{d}{dt}(AC) = \frac{dA}{dt}C$$

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Now that we have derivatives we can start looking at our systems of linear equations differently.

$$\begin{aligned} x_1' &= p_{11}(t)x_1 + p_{12}(t)x_2 + \cdots + p_{1n}(t)x_n + f_1(t) \\ x_2' &= p_{21}(t)x_1 + p_{22}(t)x_2 + \cdots + p_{2n}(t)x_n + f_2(t) \\ &\vdots \\ x_n' &= p_{n1}(t)x_1 + p_{n2}(t)x_2 + \cdots + p_{nn}(t)x_n + f_n(t) \end{aligned}$$

If we create a matrix $P(t) = [p_{ij}(t)]$ of the coefficients in the first order system above and a column vectors $\vec{f}(t) = [f_i(t)]$, $\vec{x}(t) = [x_i]$ we can write our system in the form

$$\begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} p_{11}(t) & p_{12}(t) & \cdots & p_{1n}(t) \\ p_{21}(t) & p_{22}(t) & \cdots & p_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1}(t) & p_{n2}(t) & \cdots & p_{nn}(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

or

$$\frac{d\vec{x}(t)}{dt} = P(t)\vec{x}(t) + \vec{f}(t)$$

Section 5.1 Matrices and Linear Systems

If we create a matrix $P(t) = [p_{ij}(t)]$ of the coefficients in the first order system above and a column vectors $\vec{f}(t) = [f_i(t)]$, $\vec{x}(t) = [x_i]$ we can write our system in the form

$$\begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} p_{11}(t) & p_{12}(t) & \cdots & p_{1n}(t) \\ p_{21}(t) & p_{22}(t) & \cdots & p_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ p_{n1}(t) & p_{n2}(t) & \cdots & p_{nn}(t) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{bmatrix}$$

$$\frac{d\vec{x}(t)}{dt} = P(t)\vec{x}(t) + \vec{f}(t)$$

Consider the system

$$\begin{aligned} x_1' &= 4x_1 - 3x_2 \\ x_2' &= 6x_1 - 7x_2 \end{aligned}$$

$x = x_1$
 $3x_2 = x_1'$
 $x'' + (x) + x = 0$

$$\rightarrow x_1 = 3e^{2t}, x_2 = 2e^{2t}$$

or

$$\rightarrow x_1 = e^{-5t}, x_2 = 3e^{-5t}$$

$$\vec{x}_1 = \begin{bmatrix} x_{11} \\ x_{12} \end{bmatrix}$$

$$\vec{x}_2 = \begin{bmatrix} x_{21} \\ x_{22} \end{bmatrix}$$

What will this look like in matrix form?

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\frac{d\vec{x}}{dt} = \vec{x}' = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \vec{x}$$

Let's verify that $\vec{x}_1(t) = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}$ and $\vec{x}_2(t) = \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix}$ are **solution vectors** to the system

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$\begin{bmatrix} 6e^{2t} \\ 4e^{2t} \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}$$

$$\begin{bmatrix} 6e^{2t} \\ 4e^{2t} \end{bmatrix} = \begin{bmatrix} 4(3e^{2t}) - 3(2e^{2t}) \\ 6(3)e^{2t} - 7(2)e^{2t} \end{bmatrix} = \begin{bmatrix} 12e^{2t} - 6e^{2t} \\ 18e^{2t} - 14e^{2t} \end{bmatrix} = \begin{bmatrix} 6e^{2t} \\ 4e^{2t} \end{bmatrix}$$

$$\vec{x}_2' = \begin{bmatrix} -5e^{-5t} \\ -15e^{-5t} \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix} \begin{bmatrix} e^{-5t} \\ 3e^{-5t} \end{bmatrix} = \begin{bmatrix} 4e^{-5t} - 9e^{-5t} \\ 6e^{-5t} - 21e^{-5t} \end{bmatrix} = \begin{bmatrix} -5e^{-5t} \\ -15e^{-5t} \end{bmatrix}$$

Section 5.1 Matrices and Linear Systems

Theorem: Let $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$ be n solutions of the homogeneous linear equation *system*

$$\frac{d\vec{x}(t)}{dt} = P(t)\vec{x}(t)$$

on an open interval I . If $c_1, c_2, \dots, c_n$ are constants then the linear combination

$$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + \dots + c_n\vec{x}_n$$

is also a solution.

We recall that before we also knew we had all of the solutions to an n^{th} order homogeneous equation when we had n linearly independent ones. Given the fact that we can transform an n^{th} order equation to a system of n equations we expect the same to be true here, indeed it is. Here we will have n solution vectors.

The vectors $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$ are linearly independent if

$$c_1\vec{x}_1 + c_2\vec{x}_2 + \dots + c_n\vec{x}_n = 0$$

Implies that the constants $c_1 = c_2 = \dots = c_n = 0$

Theorem Wronskians of Solutions

Suppose that $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$ are n solutions of the homogeneous linear equation $\vec{x}' = P(t)\vec{x}$ on an open interval I and that $P(t)$ is continuous on I . Let W be the determinant of the matrix formed taking the vectors $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$ as columns,

$$W = |\vec{x}_1 \ \vec{x}_2 \ \dots \ \vec{x}_n|$$

Then if $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$ are linearly **dependent** on I then $W=0$ at each point of I .

If $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$ are linearly **independent** on I , $W \neq 0$ at each point of I .

While this may seem different than the original wronskian we have used in the past for n^{th} order equations as they involved derivatives of the n solutions, but recall that our linear systems of equations can come from transformed n^{th} order equations.

$$x^{(n)} = f(t, x, x', x'', \dots, x^{(n-1)}) \quad \leftarrow \quad x_1 = x, \ x_2 = x', \ x_3 = x'', \ \dots, \ x_n = x^{(n-1)}$$

So the components of a solution vector $\vec{x}_i = [x_1, x_2, x_3, \dots, x_n] = [x, x', x'', x^{(3)}, \dots, x^{(n-1)}]$ taken as a column make up the same structure as our previous representation of the wronskian.

Section 5.1 Matrices and Linear Systems

Consider the system of equations

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix} \vec{x}$$

It has solution vectors $\vec{x}_1 = \begin{bmatrix} 2e^t \\ 2e^t \\ e^t \end{bmatrix}$, $\vec{x}_2 = \begin{bmatrix} 2e^{3t} \\ 0 \\ -e^{3t} \end{bmatrix}$, $\vec{x}_3 = \begin{bmatrix} 2e^{5t} \\ -2e^{5t} \\ e^{5t} \end{bmatrix}$

Let's check their linear independence..

W

$$\begin{vmatrix} 2e^t & 2e^{3t} & 2e^{5t} \\ -2e^t & 0 & -2e^{5t} \\ e^t & -e^{3t} & e^{5t} \end{vmatrix} = -2e^t \begin{vmatrix} 2e^{3t} & 2e^{5t} \\ -e^{3t} & e^{5t} \end{vmatrix} + 0 + 2e^{5t} \begin{vmatrix} 2e^t & 2e^{3t} \\ -2e^t & 0 \end{vmatrix}$$

$$= -2e^t (2e^{8t} - 2e^{8t}) + 2e^{5t} (-2e^{4t} - 2e^{4t}) \neq 0$$

THEOREM 3 General Solutions of Homogeneous Systems

Let $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$ be n linearly independent solutions of the homogeneous linear equation $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on an open interval I , where $\mathbf{P}(t)$ is continuous. If $\mathbf{x}(t)$ is any solution whatsoever of the equation $\mathbf{x}' = \mathbf{P}(t)\mathbf{x}$ on I , then there exist numbers $c_1, c_2, \dots, c_n$ such that

$$\mathbf{x}(t) = c_1 \mathbf{x}_1(t) + c_2 \mathbf{x}_2(t) + \dots + c_n \mathbf{x}_n(t) \quad (35)$$

for all t in I .

Section 5.1 Matrices and Linear Systems

Initial Value Problems and Elementary Row Operations

Consider the system of equations:

$$\begin{aligned} x_1' &= 3x_1 - 2x_2 \\ x_2' &= -x_1 + 3x_2 - 2x_3 \\ x_3' &= -x_2 + 3x_3 \end{aligned} \quad \text{Same as} \quad \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \text{If we call } \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \frac{d\vec{x}}{dt} \text{ and } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \vec{x}(t)$$

We have

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix} \vec{x}(t) \quad \text{with solution vectors (we will learn to find these)} \quad \vec{x}_1 = \begin{bmatrix} 2e^t \\ 2e^t \\ e^t \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 2e^{3t} \\ 0 \\ -e^{3t} \end{bmatrix}, \vec{x}_3 = \begin{bmatrix} 2e^{5t} \\ -2e^{5t} \\ e^{5t} \end{bmatrix}$$

So by our previous theorem the full solution is

$$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3 = c_1 \begin{bmatrix} 2e^t \\ 2e^t \\ e^t \end{bmatrix} + c_2 \begin{bmatrix} 2e^{3t} \\ 0 \\ -e^{3t} \end{bmatrix} + c_3 \begin{bmatrix} 2e^{5t} \\ -2e^{5t} \\ e^{5t} \end{bmatrix}$$

Which could also be written in the form:

$$\vec{x}(t) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2e^t & 2e^{3t} & 2e^{5t} \\ 2e^t & 0 & -2e^{5t} \\ e^t & -e^{3t} & e^{5t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = X(t)\vec{c}$$

Or also in the form

$$\begin{aligned} x_1(t) &= c_1 2e^t + c_2 2e^{3t} + c_3 2e^{5t} \\ x_2(t) &= c_1 2e^t + c_2 0 + c_3 (-2e^{5t}) \\ x_3(t) &= c_1 e^t + c_2 (-e^{3t}) + c_3 e^{5t} \end{aligned}$$

If we were given some initial data we could write this in the following forms.

$$\begin{aligned} x_1(0) &= 0 \\ x_2(0) &= 2 \\ x_3(0) &= 6 \end{aligned} \quad \text{or} \quad \vec{x}(0) = \begin{bmatrix} 0 \\ 2 \\ 6 \end{bmatrix}$$

So to find our constants solving either of the equivalent systems for c_1, c_2, c_3

$$\begin{aligned} x_1(0) &= c_1 2e^0 + c_2 2e^0 + c_3 2e^0 = 0 \\ x_2(0) &= c_1 2e^0 + c_2 0 + c_3 (-2e^0) = 2 \\ x_3(0) &= c_1 e^0 + c_2 (-e^0) + c_3 e^0 = 6 \end{aligned} \quad \vec{x}(0) = \begin{bmatrix} 2e^0 & 2e^0 & 2e^0 \\ 2e^0 & 0 & -2e^0 \\ e^0 & -e^0 & e^0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 6 \end{bmatrix}$$

$$\begin{aligned} 2c_1 + 2c_2 + c_3 &= 0 \\ 2c_1 - 2c_3 &= 2 \\ c_1 - c_2 + c_3 &= 6 \end{aligned} \quad \begin{bmatrix} 2 & 2 & 2 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 6 \end{bmatrix}$$

Section 5.1 Matrices and Linear Systems

You can solve these systems a number of ways, substitution with the equations, elimination with the equations or matrices or inverses with the matrices.

$$\begin{bmatrix} 2 & 2 & 2 \\ 2 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 6 \end{bmatrix}$$

Let's review how we do this with elimination with matrices.

The Augmented Matrix

Elimination can be tedious and so if we can reduce the monotony a bit we should. One way is stop writing each variable every step. Instead we will use an array of numbers where the position of that number assigns it as a coefficient of a specific variable or constant.

$$\begin{aligned} 2c_1 + 2c_2 + c_3 &= 0 \\ 2c_1 - 2c_3 &= 2 \\ c_1 - c_2 + c_3 &= 6 \end{aligned}$$



$$\left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 2 & 0 & -2 & 2 \\ 1 & -1 & 1 & 6 \end{array} \right]$$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_{11} \\ a_{21} & a_{22} & \cdots & a_{2n} & b_{21} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_{m1} \end{array} \right]$$

Anatomy of an augmented matrix

With augmented matrices we perform **elementary row operations** just like the elementary operation but on rows in our matrix instead of on equations.

1. Interchange any two rows.
2. Multiply any row by a nonzero constant.
3. Add a multiple of one equation to another.

Gaussian Elimination

$$\left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 2 & 0 & -2 & 2 \\ 1 & -1 & 1 & 6 \end{array} \right] \xrightarrow{\text{Row operations to reduced row echelon form}} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 4 & 4 \end{array} \right] \xrightarrow{\text{Back to system}} \begin{aligned} c_1 + c_2 + c_3 &= 0 \\ c_2 + 2c_3 &= -1 \\ 4c_3 &= 4 \end{aligned} \xrightarrow{\text{Back solve}} \begin{aligned} c_1 &= 2 \\ c_2 &= -3 \\ c_3 &= 1 \end{aligned}$$

Typical Procedure:

1. If it makes the things easier interchange rows (If a row has a 1 in first column make it the first row)
2. Use Row 1 to eliminate $a_{12} \cdots a_{1n}$ (get all zeros below a_{11} in the first column)
3. Use Row 2 to eliminate $a_{23} \cdots a_{2n}$ (ger all zeros below a_{22} in the second column)
4. Continue this approach working from left to right until you are in rref form.

Once we have the constants we have our solution!

$$\vec{x}(t) = 2 \begin{bmatrix} 2e^t \\ 2e^t \\ e^t \end{bmatrix} - 3 \begin{bmatrix} 2e^{3t} \\ 0 \\ -e^{3t} \end{bmatrix} + 1 \begin{bmatrix} 2e^{5t} \\ -2e^{5t} \\ e^{5t} \end{bmatrix} = \begin{bmatrix} 4e^2 - 6e^{3t} + 2e^{5t} \\ 4e^t - 2e^{5t} \\ 2e^t + 3e^{3t} + e^{5t} \end{bmatrix}$$

Section 5.1 Matrices and Linear Systems

Let's go through the procedure

$$\left[\begin{array}{ccc|c} 2 & 2 & 2 & 0 \\ 2 & 0 & -2 & 2 \\ 1 & -1 & 1 & 6 \end{array} \right]$$

Section 1.2 Linear Systems and Matrices

Computers can do elimination, however make sure you can too! Chances are an exam or quiz problem will require hand calculations!

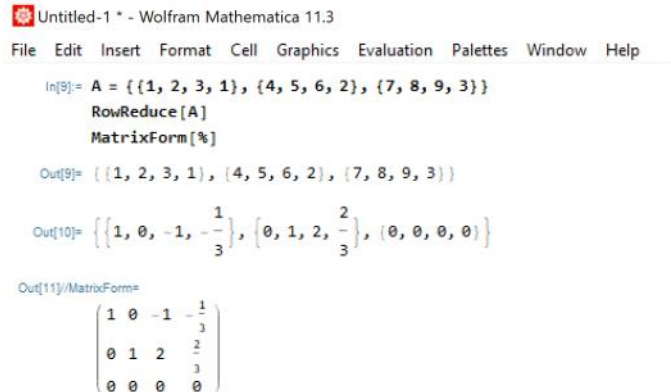
Use technology to find the solution to the system

$$\begin{aligned}x_1 + 2x_2 + 3x_3 &= 1 \\4x_1 + 5x_2 + 6x_3 &= 2 \\7x_1 + 8x_2 + 9x_3 &= 3\end{aligned}$$

Mathematica

```
A={{1,2,3,1},{4,5,6,2},{7,8,9,3}}
RowReduce[A]
MatrixForm[%]
```

To execute the code above hit shift+enter



```
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File Edit Insert Format Cell Graphics Evaluation Palettes Window Help

In[8]:= A = {{1, 2, 3, 1}, {4, 5, 6, 2}, {7, 8, 9, 3}}
RowReduce[A]
MatrixForm[%]

Out[9]= {{1, 2, 3, 1}, {4, 5, 6, 2}, {7, 8, 9, 3}}

Out[10]= {{1, 0, -1, -1/3}, {0, 1, 2, 2/3}, {0, 0, 0, 0}}

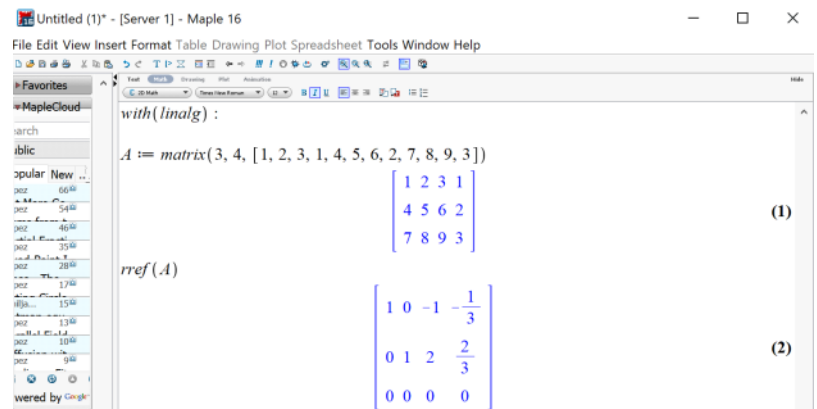
Out[11]= MatrixForm=

$$\begin{pmatrix} 1 & 0 & -1 & -\frac{1}{3} \\ 0 & 1 & 2 & \frac{2}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

```

Maple

```
with(linalg)
A:=matrix(3,4,[1,2,3,1,4,5,6,2,7,8,9,3])
rref(A)
```



```
Untitled (1)* - [Server 1] - Maple 16
File Edit View Insert Format Table Drawing Plot Spreadsheet Tools Window Help

with(linalg):
A := matrix(3, 4, [1, 2, 3, 1, 4, 5, 6, 2, 7, 8, 9, 3])
rref(A)
```

(1)

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 2 \\ 7 & 8 & 9 & 3 \end{pmatrix}$$

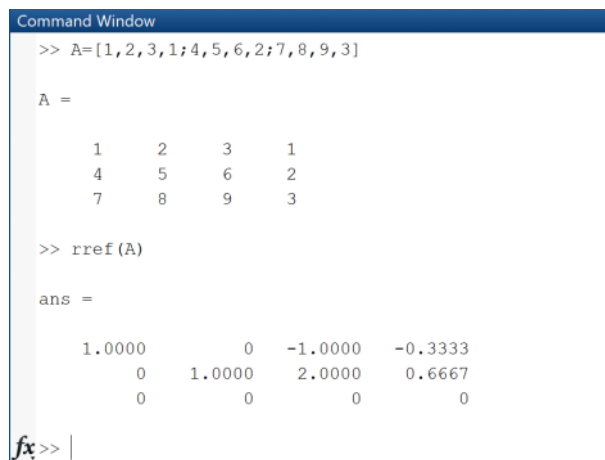
(2)

$$\begin{pmatrix} 1 & 0 & -1 & -\frac{1}{3} \\ 0 & 1 & 2 & \frac{2}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Matlab or Octave (Octave is free!)

```
A=[1,2,3,1; 4,5,6,2 ; 7,8,9,3]
Rref(A)
```

You can install octave on your machine or take advantage of <https://octave-online.net/>



```
Command Window
>> A=[1,2,3,1;4,5,6,2;7,8,9,3]

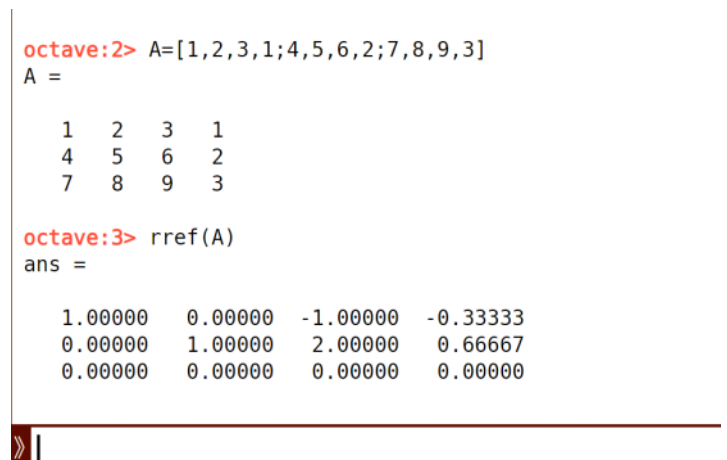
A =

     1     2     3     1
     4     5     6     2
     7     8     9     3

>> rref(A)

ans =

    1.0000     0    -1.0000   -0.3333
         0     1     2.0000    0.6667
         0     0         0         0
```



```
octave:2> A=[1,2,3,1;4,5,6,2;7,8,9,3]
A =

     1     2     3     1
     4     5     6     2
     7     8     9     3

octave:3> rref(A)
ans =

    1.00000    0.00000   -1.00000   -0.33333
    0.00000    1.00000    2.00000    0.66667
    0.00000    0.00000    0.00000    0.00000
```

Section 5.1 Matrices and Linear Systems

Non Homogeneous Solutions

Suppose we want to solve the nonhomogeneous case

$$\frac{d\vec{x}}{dt} = P(t)\vec{x} + \vec{f}(t)$$

Just like before we need to find a $\vec{x}_c, \vec{x}_p$ but this time each is a vector solution we find $\vec{x}_c$ by solving

$$\frac{d\vec{x}}{dt} = P(t)\vec{x}$$

And $\vec{x}_p$ by solving

$$\frac{d\vec{x}}{dt} = P(t)\vec{x} + \vec{f}(t)$$

Then the full solution is again $\vec{x}(t) = \vec{x}_c + \vec{x}_p$

Let's modify our previous problem to be:

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} -9t + 13 \\ 7t - 15 \\ -6t + 7 \end{bmatrix}$$

From what we have found before:

$$\vec{x}_c = \begin{bmatrix} 4e^2 - 6e^{3t} + 2e^{5t} \\ 4e^t - 2e^{5t} \\ 2e^t + 3e^{3t} + e^{5t} \end{bmatrix}$$

It turns out that (we don't know how to find this yet though) $\vec{x}_p = \begin{bmatrix} 3t \\ 5 \\ 2t \end{bmatrix}$ It will be a quiz problem to verify this.... You can do so by plugging it in to the DE and showing that:

$$\begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix} \begin{bmatrix} 3t \\ 5 \\ 2t \end{bmatrix} + \begin{bmatrix} -9t + 13 \\ 7t - 15 \\ -6t + 7 \end{bmatrix}$$

Once verified we can say the full solution is.

$$\vec{x}(t) = \vec{x}_c + \vec{x}_p = \begin{bmatrix} 4e^2 - 6e^{3t} + 2e^{5t} - 9t + 13 \\ 4e^t - 2e^{5t} + 7t - 15 \\ 2e^t + 3e^{3t} + e^{5t} + 2t \end{bmatrix}$$