

Section 5.2 The Eigen Value Method for Homogeneous Systems

Let's take a look at the general form of n first order homogeneous differential equations with constant coefficients and try to think about what a solution might look like for such a system.

$$\begin{aligned}x_1' &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\x_2' &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\&\vdots \\x_n' &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n\end{aligned}$$

In words we have that the first derivative of each unknown function x_i is a sum of multiples of the other unknown functions and itself. Before when we looked at the following differential equation

$$x' = kx$$

We said that the solution is likely something like $x = e^{kt}$ because then $x' = ke^{kt} = kx$

We can extend this idea to the system of n first order equation by also assuming that each unknown function is likely an exponential. Perhaps each with a different coefficient in front of it.

For example if $x_1(t) = v_1e^{\lambda t}$ and then $x_2 = v_2e^{\lambda t}$ and $\cdots x_n = v_ne^{\lambda t}$ we could look at

$$x_1' = a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n$$

Which would become $\lambda v_1e^{\lambda t} = a_1v_1e^{\lambda t} + a_2v_2e^{\lambda t} + \cdots + a_nv_ne^{\lambda t}$

$$\text{or } \lambda v_1 = a_1v_1 + a_2v_2 + \cdots + a_nv_n$$

So if I can find $\lambda, v_1, \dots, v_n$ which make the above hold I would know that I have a solution for $x_1, x_2, \dots, x_n$ and indeed we can since if I apply the same assumption to each of the other n equations I would end up with

$$\begin{aligned}\lambda v_1 &= a_1v_1 + a_2v_2 + \cdots + a_nv_n \\ \lambda v_2 &= a_1v_1 + a_2v_2 + \cdots + a_nv_n \\ &\vdots \\ \lambda v_n &= a_1v_1 + a_2v_2 + \cdots + a_nv_n\end{aligned}$$

A system of n equations with n unknowns ($v_1, v_2, \dots, v_n$) provided we know λ . If I do know lambda, then I can solve to get a solution to the system of the form:

$$\vec{x}_1 = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} v_1e^{\lambda t} \\ v_2e^{\lambda t} \\ \vdots \\ v_ne^{\lambda t} \end{bmatrix}$$

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$$\vec{x}_1 = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} v_1 e^{\lambda t} \\ v_2 e^{\lambda t} \\ \vdots \\ v_n e^{\lambda t} \end{bmatrix}$$

Theorem: Let $\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n$ be n solutions of the homogeneous linear system of equations

$$\frac{d\vec{x}(t)}{dt} = P(t)\vec{x}(t)$$

on an open interval I . If $c_1, c_2, \dots, c_n$ are constants then the linear combination

$$\vec{x}(t) = c_1 \vec{x}_1 + c_2 \vec{x}_2 + \dots + c_n \vec{x}_n$$

is also a solution.

Further recall our theorem from section 5.1 we know the full solution to the system of equations is actually a linear combination of vectors of the form we have worked out, and we know there will be n linearly independent ones. This suggests there may be other lambdas and other sets of equations to solve for each of the possible combinations of possible $x_1, \dots, x_n$ that also solve the system that can make up the rest of the solution vectors. So how do we find them?

First let's put this system in to matrix form:

$$\begin{aligned} x_1' &= a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n \\ x_2' &= a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ &\vdots \\ x_n' &= a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n \end{aligned} \quad \longrightarrow \quad \begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \longrightarrow \quad \vec{x}' = A\vec{x}$$

Let's assume our solution vector is of the form

$$\vec{x} = \begin{bmatrix} v_1 e^{\lambda t} \\ v_2 e^{\lambda t} \\ \vdots \\ v_n e^{\lambda t} \end{bmatrix} \quad \text{Then } \vec{x}' = \begin{bmatrix} \lambda v_1 e^{\lambda t} \\ \lambda v_2 e^{\lambda t} \\ \vdots \\ \lambda v_n e^{\lambda t} \end{bmatrix} = \lambda \begin{bmatrix} v_1 e^{\lambda t} \\ v_2 e^{\lambda t} \\ \vdots \\ v_n e^{\lambda t} \end{bmatrix}$$

This means that

$$\begin{bmatrix} x_1' \\ x_2' \\ \vdots \\ x_n' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \quad \longrightarrow \quad \begin{bmatrix} v_1 e^{\lambda t} \\ v_2 e^{\lambda t} \\ \vdots \\ v_n e^{\lambda t} \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \lambda \begin{bmatrix} v_1 e^{\lambda t} \\ v_2 e^{\lambda t} \\ \vdots \\ v_n e^{\lambda t} \end{bmatrix}$$

$$\text{Or that } \vec{x}' = A\vec{x} = \lambda\vec{x}$$

$A\vec{x} = \lambda\vec{x}$ is what we call the **eigen value problem!**

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$A\vec{x} = \lambda\vec{x}$ eigen value problem

The general eigen value problem comes up everywhere in mathematics but here it means that the action of the matrix A on the vector $\vec{x}$ gives us a scalar times $\vec{x}$. We call the scalar and vectors that satisfy this condition the **eigen value** and **eigen vector**. They are heckin important.

So how do we find eigen values and eigen vectors? Well a quick re-write of the equation above gives us

or
$$A\vec{x} - \lambda\vec{x} = 0$$

$$A\vec{x} - I\lambda\vec{x} = (A - \lambda I)\vec{x} = 0$$

So we want to solve the problem $(A - \lambda I)\vec{x} = 0$ for $\lambda, \vec{x}$

You may recall having been told that for a problem like $M\vec{x} = \vec{b}$ where $\vec{b} \neq 0$ will only have a unique solution if $\det(M) \neq 0$ and this is true, but if we want $M\vec{x} = 0$ to have solutions $\vec{x} \neq 0$ our requirement is the opposite, the matrix M in this case will need to have $\det(M) = 0$. If it were not zero the only solution to $M\vec{x} = 0$ would be the trivial solution $\vec{x} = 0$.

This means that we will only have non-trivial solutions if

$$\det(A - \lambda I) = 0$$

When we take this determinant we will get a polynomial in lambda equal to zero for which we will find the roots, this is just like our characteristic equation for n^{th} order homogeneous equations, infact we call it the characteristic equation.

So if we want to find our eigen values to get the n lambdas that define the exponentials of our solutions $e^{\lambda t}$ we are tasked with taking the determinant of the following matrix

$$\begin{aligned} & \det \left(\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} -1/2 \\ -1 \\ 1 \end{bmatrix} e^{-3t} - \lambda \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \right) \\ &= \det \left(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} - \begin{bmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \lambda \end{bmatrix} \right) \\ &= \det \left(\begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{bmatrix} \right) \end{aligned}$$

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Once we have the eigen values we are then tasked with finding the eigen vectors. If lambda is known this means solving the system

$$\left(\begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{bmatrix} \right) \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = 0$$

For $v_1, v_2, \dots, v_n$

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$$

Let's find the eigen values and eigen vectors of the following matrix.

So we get $\lambda_1 = 1, \lambda_2 = 2$ So then...

$$\begin{aligned} (A - 2I) &= \begin{bmatrix} 2-2 & 1 \\ 0 & 1-2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \\ (A - 1I) &= \begin{bmatrix} 2-1 & 1 \\ 0 & 1-1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \end{aligned}$$

Let's find the eigen vectors.

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So we have $\lambda_1 = 1$, $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\lambda_2 = 2$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Ok great, but how does this give us the solution to a system of differential equations. Well let's look at the system of equations.

$$\begin{aligned}x_1' &= 2x_1 + x_2 \\x_2' &= x_1\end{aligned}$$

Which can be written as

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

And we would guess solutions $\vec{x} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{\lambda t}$

Which when plugged into the matrix form of the equation gives us

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \lambda e^{\lambda t} = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} e^{\lambda t}$$

Which is exactly the problem

$$\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \lambda = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} - \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \longrightarrow \left(\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} - \lambda I \right) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

Which we just solved. So we get two solutions to the system of equations based on our eigen values and eigen vectors.

$$\vec{x}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^t, \vec{x}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t}$$

Or a full general solution which is a linear combination of the two given by

$$\vec{x} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^t + c_2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} e^{2t}$$

Or denoting the eigen vectors as $\vec{v}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, and eigen values as $\lambda_1 = 1$, $\lambda_2 = 2$

$$\vec{x} = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t}$$

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In general for a system of n first order homogeneous equations

$$\begin{aligned}x_1' &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \\x_2' &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \\&\vdots \\x_n' &= a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n\end{aligned}$$

For which the coefficient matrix has **n distinct real eigen values $\lambda_1, \lambda_2, \dots, \lambda_n$** which implies we also have n distinct eigen vectors $\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n$ the general solution will be.

$$\vec{x}(t) = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t} + \cdots + c_n \vec{v}_n e^{\lambda_n t}$$

Example: Solve the system of differential equations given by:

$$\begin{aligned}x_1' &= x_1 - 2x_2 + 4x_3 \\x_2' &= -2x_2 + x_3 \\x_3' &= x_2 - 2x_3\end{aligned} \quad \longrightarrow \quad \begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \quad \longrightarrow \quad \vec{x}' = A\vec{x}$$

For this problem our steps will be the following:

Step 1: Calculate $\det(A - \lambda I)$ set it equal to zero and solve the resulting polynomial for lambda, that is solve the equation:

$$\det(A - \lambda I) = 0$$

Step 2: Take the resulting λ 's and get the matrices $A - \lambda I$ then find the vectors for which

$$(A - \lambda I)\vec{v} = 0$$

Step 3: If there are n distinct real eigen values build your solution as

$$\vec{x}(t) = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t} + \cdots + c_n \vec{v}_n e^{\lambda_n t}$$

Step 4: If given initial data $x_1(a) = b_1, x_2(a) = b_2, \dots, x_n(a) = b_n$ find $c_1, c_2, \dots, c_n$ by solving the equation

$$X(a)\vec{c} = \begin{bmatrix} \vec{v}_1 e^{\lambda_1 a} & \vec{v}_2 e^{\lambda_2 a} & \cdots & \vec{v}_n e^{\lambda_n a} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

Where the columns of the matrix X are formed taking the the eigenvectors times their exponentials as columns.

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Step 1. Find the characteristic polynomial and its zeros will be the eigen values.

$$\det \begin{pmatrix} 1-\lambda & 2 & 4 \\ 0 & -2-\lambda & 1 \\ 0 & 1 & -2-\lambda \end{pmatrix} = 0$$

$$(1-\lambda) \begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} + 0 \begin{vmatrix} 2 & 4 \\ 1 & -2-\lambda \end{vmatrix} + 0 \begin{vmatrix} 2 & 4 \\ -2-\lambda & 1 \end{vmatrix} = 0$$

$$(1-\lambda)((-2-\lambda)(-2-\lambda) - 1) = 0$$

$$(1-\lambda)(\lambda^2 + 4\lambda + 4 - 1) = (1-\lambda)(\lambda^2 + 4\lambda + 3) = (1-\lambda)(\lambda+1)(\lambda+3) = 0$$

$$\lambda_1 = 1 \quad \lambda_2 = -1 \quad \lambda_3 = -3$$

Step 2. Find the null spaces of the matrices $(A - \lambda_i I)$ those are your eigen vectors, each associated with an eigen value λ_i

$$(A - 1I)\vec{x} = \begin{bmatrix} 0 & 2 & 4 \\ 0 & -3 & 1 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad (A - (-1)I)\vec{x} = \begin{bmatrix} 2 & 2 & 4 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad (A - (-3)I)\vec{x} = \begin{bmatrix} 4 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

octave:4> rref(A-eye(3))
ans =

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 &= s \\ x_2 &= 0 \\ x_3 &= 0 \end{aligned}$$

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} s, \lambda_1 = 1$$

octave:5> rref(A-(-1)*eye(3))
ans =

$$\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_3 &= s \\ x_2 - x_3 &= 0 \rightarrow x_2 = s \\ x_1 + 3x_3 &= 0 \rightarrow x_1 = -3s \end{aligned}$$

$$\vec{v}_2 = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} s, \lambda_2 = -1$$

octave:8> rref(A-(-3)*eye(3))
ans =

$$\begin{bmatrix} 1 & 0 & 1/2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_3 &= s \\ x_2 + x_3 &= 0 \rightarrow x_2 = -s \\ x_1 + \frac{1}{2}x_3 &= 0 \rightarrow x_1 = -\frac{1}{2}s \end{aligned}$$

$$\vec{v}_3 = \begin{bmatrix} -\frac{1}{2} \\ -1 \\ 1 \end{bmatrix} s, \lambda_3 = -3$$

Take $s = 1$ for simplicity and then the solution is:

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$$\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} e^{-t} + c_3 \begin{bmatrix} -1/2 \\ -1 \\ 1 \end{bmatrix} e^{-3t}$$

Let's suppose we were given the initial data $x_1(0) = 1, x_2(0) = 1, x_3(0) = 3$

Let's set up the system we need to solve to find c_1, c_2, c_3 and then ask a computer to solve it!

So then then the solution to the initial value problem is
$$\vec{x}(t) = \frac{15}{2} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} e^t + 2 \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} e^{-t} + 1 \begin{bmatrix} -1/2 \\ -1 \\ 1 \end{bmatrix} e^{-3t}$$

Or equivalently

$$x_1(t) =$$

$$x_2(t) =$$

$$x_3(t) =$$

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Distinct Complex Eigen Values

Since we find our eigen values solving the equation $\det(A - \lambda I) = 0$ which gives us a polynomial in λ to find the roots of it is entirely possible to get complex values. So we should want to know how to handle them!

We will focus on the case where the coefficient matrix A has only real entries which will mean that any complex eigen values we get will come in complex conjugate pairs. $a + bi, a - bi$

Let's suppose we get a complex eigen value $\lambda = a + bi$ this means that the equation

$$A\vec{v} = (a + bi)\vec{v}$$

Since we are assuming that A has real entries it must be that there are other complex numbers in $\vec{v}$.

So we are looking at an equation of the form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} a_1 + b_1 i \\ a_2 + b_2 i \\ \vdots \\ a_n + b_n i \end{bmatrix} = (a + bi) \begin{bmatrix} a_1 + b_1 i \\ a_2 + b_2 i \\ \vdots \\ a_n + b_n i \end{bmatrix}$$

Note that if we take the complex conjugate of the whole thing $A\vec{v} = (a + bi)\vec{v}$

So we have $A\vec{\bar{v}} = (a - bi)\vec{\bar{v}}$ which directly implies the eigen vector associated with the other eigen value $a - bi$ is just the complex conjugate of the one associated with $a + bi$! This can save some work!

So we have $\lambda_1 = a + bi$ with eigen vector $\vec{v}_1 = \vec{a} + \vec{b}i$ and eigen value $\lambda_2 = a - bi$ with eigen vector $\vec{v}_2 = \vec{a} - \vec{b}i$

$$\text{Where } \vec{a} + \vec{b}i = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + i \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

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So following the form of solution before from these two eigen values we might expect two solutions of the form

$$\vec{x}_1 = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t}$$

$$\vec{x} = c_1 (\vec{a} + \vec{b}i) e^{(a+bi)t} + c_2 (\vec{a} - \vec{b}i) e^{(a-bi)t}$$

Or

$$\vec{x} = e^{at} \left(c_1 (\vec{a} + \vec{b}i) (\cos(bt) + i \sin(bt)) + c_2 (\vec{a} - \vec{b}i) (\cos(bt) - i \sin(bt)) \right)$$

But we can write this as

$$\vec{x}(t) = e^{at} \left[[(c_1 + c_2) \vec{a} \cos(bt) - (c_1 - c_2) \vec{b} \sin(bt)] + i [(c_1 - c_2) \vec{b} \cos(bt) + (c_1 + c_2) \vec{a} \sin(bt)] \right]$$

Or if we let $A = c_1 + c_2$, $B = c_1 - c_2$ and factor them out we get.

$$\vec{x}(t) = e^{at} \left[A (\vec{a} \cos(bt) - \vec{b} \sin(bt)) + Bi (\vec{b} \cos(bt) + \vec{a} \sin(bt)) \right]$$

Just as before if we use real initial data A,B will be such that we get real numbers in front of our two terms. So we can just assume real solutions anyway. Our two solutions coming from the complex eigen values is then

$$\vec{x}_1 = e^{at} (\vec{a} \cos(bt) - \vec{b} \sin(bt))$$

$$\vec{x}_2 = e^{at} (\vec{b} \cos(bt) + \vec{a} \sin(bt))$$

So that our contribution to the system is just as above but without the "i"

$$\vec{x}(t) = e^{at} \left[A (\vec{a} \cos(bt) - \vec{b} \sin(bt)) + B (\vec{b} \cos(bt) + \vec{a} \sin(bt)) \right]$$

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Also note that if we had only taken one of the eigen vectors and eigen values

$$\begin{aligned}\vec{x}(t) &= (\vec{a} + \vec{b}i) e^{(a+bi)t} = (\vec{a} + \vec{b}i) (\cos(bt) + i\sin(bt)) \\ &= e^{at} (\vec{a} \cos(bt) - \vec{b} \sin(bt)) + ie^{at} (\vec{b} \cos(bt) + \vec{a} \sin(bt))\end{aligned}$$

The same form we had from simplifying before. So the complex conjugate pairs both suggest the same two solutions. This means that we can avoid memorizing the form of solutions before and just take one of the eigen values and vectors to find our contribution to the solution made by the complex eigen values.

We do it by

1. finding the eigen vector associated with one of the eigen values
2. build a solution as $\vec{x}(t) = (\vec{a} + \vec{b}i) (\cos(bt) + i\sin(bt))$
3. Separate out the real and imaginary parts. $\vec{x}_1(t) = \text{Re}(\vec{x}(t)), \vec{x}_2(t) = \text{Im}(\vec{x}(t))$
4. The contribution will be $c_1\vec{x}_1 + c_2\vec{x}_2$.

Let's do an example.... Find the general solution to the system of equations

$$\begin{aligned}x_1' &= 4x_1 - 3x_2 \\ x_2' &= 3x_1 + 4x_2\end{aligned}$$

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So we get $\vec{x}(t) = c_1 e^{4t} \begin{bmatrix} \cos(3t) \\ \sin(3t) \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} -\sin(3t) \\ \cos(3t) \end{bmatrix}$

