

## Section 5.2 The Eigen Value Method for Homogeneous Systems

For fun let's look at the system 
$$\begin{aligned}x_1' &= -4x_1 - 3x_2 \\x_2' &= 3x_1 - 4x_2\end{aligned}$$

A slight modification of the system before:

$$\begin{aligned}x_1' &= 4x_1 - 3x_2 \\x_2' &= 3x_1 + 4x_2\end{aligned}$$

```
octave:12> A=[ -4 -3; 3 -4]
```

```
A =
```

```
-4 -3
 3 -4
```

Coefficient Matrix

```
octave:13> [vecs,lambda]=eig(A)
```

```
vecs =
```

```
0.7071 + 0i 0.7071 - 0i
0 - 0.7071i 0 + 0.7071i
```

Same eigen vectors, just scaled differently. Recall

before we had  $\begin{bmatrix} 1 \\ -i \end{bmatrix}, \begin{bmatrix} 1 \\ i \end{bmatrix}$

```
lambda =
```

Diagonal Matrix

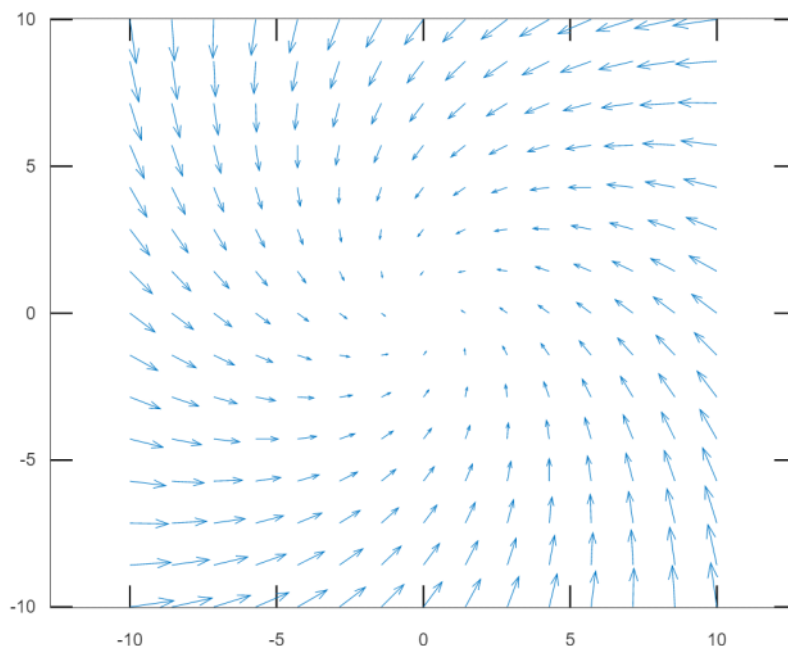
```
-4.0000 + 3.0000i 0
0 -4.0000 - 3.0000i
```

Only difference here is that the real part of the eigen values is negative!

Our solution would be

$$x(t) = e^{-4t} \begin{bmatrix} c_1 \cos(3t) - c_2 \sin(3t) \\ c_1 \sin(3t) + c_2 \cos(3t) \end{bmatrix}$$

Direction Field



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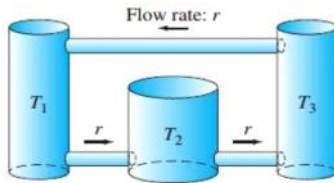


**FIGURE 5.2.5.** The three brine tanks of Example 4.

Find the amounts of salt at time  $t$  in the three brine tanks of the figure to the right if  $V_1 = 50, V_2 = 25, V_3 = 50$  and  $r = 10 \text{ gal/min}$

So our system is:

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} -0.2 & 0 & 0.2 \\ 0.2 & -0.4 & 0 \\ 0 & 0.4 & -0.2 \end{bmatrix} \vec{x}$$

Let's solve it with the help of a computer!

```
octave:17> A=[-0.2 0 0.2; 0.2 -0.4 0; 0 0.4 -0.2]
```

```
A =
```

```
-0.2000      0    0.2000
 0.2000 -0.4000      0
      0    0.4000 -0.2000
```

```
octave:18> eig(A)
```

```
ans =
```

```
-0.0000 + 0i
-0.4000 + 0.2000i
-0.4000 - 0.2000i
```

```
octave:19> [v,lambda]=eig(A)
```

```
v =
```

```
-0.6667 + 0i    0.3536 + 0.3536i    0.3536 - 0.3536i
-0.3333 + 0i    0.3536 - 0.3536i    0.3536 + 0.3536i
-0.6667 + 0i   -0.7071 + 0i   -0.7071 - 0i
```

```
lambda =
```

```
Diagonal Matrix
```

```
-0.0000 + 0i      0      0
      0 -0.4000 + 0.2000i      0
      0      0 -0.4000 - 0.2000i
```

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```
octave:19> [v,lambda]=eig(A)
```

v =

```
-0.6667 + 0i  0.3536 + 0.3536i  0.3536 - 0.3536i
-0.3333 + 0i  0.3536 - 0.3536i  0.3536 + 0.3536i
-0.6667 + 0i -0.7071 + 0i    -0.7071 - 0i
```

lambda =

Diagonal Matrix

```
-0.0000 + 0i  0  0
0 -0.4000 + 0.2000i  0
0 0 -0.4000 - 0.2000i
```

$$\vec{v}_1 = \begin{bmatrix} -\frac{2}{3} \\ -\frac{1}{3} \\ \frac{2}{3} \end{bmatrix}, \lambda_1 = 0$$

$$\vec{v}_2 = \begin{bmatrix} 0.3536 + 0.3536i \\ 0.3536 - 0.3536i \\ -0.7071 \end{bmatrix}, \lambda_2 = -0.4 + 0.2i$$

$$\vec{v}_3 = \overline{\vec{v}_2}, \quad \lambda_3 = \overline{\lambda_2}$$

Recall we can use any multiple of the eigenvectors we want so why not make them look a bit nicer!

$$\vec{v}_1 = \begin{bmatrix} -2/3 \\ -1/3 \\ 2/3 \end{bmatrix} \rightarrow \text{multiply by } -3 \rightarrow \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$$

$$\vec{v}_2 = \begin{bmatrix} 0.3536 + 0.3536i \\ 0.3536 - 0.3536i \\ -0.7071 \end{bmatrix}$$

For  $\vec{v}_2$  I would prefer less complex numbers at least. Note for a complex number with the same magnitude in the real and imaginary parts

$$(a + ai)(a - ai) = a^2 + a^2i - a^2i - (ai)^2 = 2a^2$$

$$(a - ai)(a + ai) = a^2 - a^2i - a^2i + (ai)^2 = -2a^2i$$

So if I multiply  $\vec{v}_2$  by  $0.3536 - 0.3536i$  I will end up with

$$\begin{bmatrix} 0.3536 + 0.3536i \\ 0.3536 - 0.3536i \\ -0.7071 \end{bmatrix} (0.3536 - 0.3536i) = \begin{bmatrix} 2(0.3536)^2 \\ 2(0.3536)^2i \\ -0.7071(0.3536 - 0.3536i) \end{bmatrix}$$

```
octave:7> v(:,2)*(v(2,2))
ans =
```

```
2.5000e-01 - 4.4409e-16i
-3.6082e-16 - 2.5000e-01i
-2.5000e-01 + 2.5000e-01i
```

$$\text{Or } \begin{bmatrix} 2.5 \\ -2.5i \\ -2.5 + 2.5i \end{bmatrix} \rightarrow \text{divide by } 2.5 \rightarrow \begin{bmatrix} 1 \\ -i \\ -1 + i \end{bmatrix} = \vec{v}_2 \quad \text{Then } \vec{v}_3 = \overline{\vec{v}_2} = \begin{bmatrix} 1 \\ i \\ -1 - i \end{bmatrix}$$

$$\text{So we have: } \vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \lambda_1 = 0, \vec{v}_2 = \begin{bmatrix} 1 \\ -i \\ -1 + i \end{bmatrix}, \lambda_2 = -0.4 + 0.2i, \vec{v}_3 = \begin{bmatrix} 1 \\ i \\ -1 - i \end{bmatrix}, \lambda_3 = -0.4 - 0.2i$$

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$$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \lambda_1 = 0, \vec{v}_2 = \begin{bmatrix} 1 \\ -i \\ -1 + 1 \end{bmatrix}, \lambda_2 = -0.4 + 0.2i, \vec{v}_3 = \begin{bmatrix} 1 \\ i \\ -1 - i \end{bmatrix}, \lambda_3 = -0.4 - 0.2i$$

From  $\vec{v}_1, \lambda_1$  we get the contribution to the solution of  $\vec{x}_1 = e^{0t} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

For the complex conjugate pairs we need only calculate one and separate the real and imaginary parts to get the contribution from both.

Let's choose  $v_2, \lambda_2$

$$\vec{x}_{cpairs} = \begin{bmatrix} 1 \\ -1 \\ -1 + i \end{bmatrix} e^{(-0.4+0.2i)t} = \begin{bmatrix} 1 \\ -1 \\ -1 + i \end{bmatrix} e^{-0.4t} (\cos(0.2t) + i\sin(0.2t))$$

Distribute the exponential and trig functions

$$= \begin{bmatrix} e^{-0.4t} (\cos(0.2t) + i\sin(0.2t)) \\ -e^{-0.4t} (\cos(0.2t) + i\sin(0.2t)) \\ (-1 + i)(e^{-0.4t} (\cos(0.2t) + i\sin(0.2t))) \end{bmatrix} = \begin{bmatrix} e^{-0.4t} \cos(0.2t) + ie^{-0.4t} \sin(0.2t) \\ -e^{-0.4t} \cos(0.2t) - ie^{-0.4t} \sin(0.2t) \\ -e^{-0.4t} \cos(0.2t) - ie^{-0.4t} \sin(0.2t) + ie^{-0.4t} \cos(0.2t) - e^{-0.4t} \sin(0.2t) \end{bmatrix}$$

Group real and imaginary parts in the vector.

$$= \begin{bmatrix} e^{-0.4t} \cos(0.2t) + ie^{-0.4t} \sin(0.2t) \\ -e^{-0.4t} \cos(0.2t) - ie^{-0.4t} \sin(0.2t) \\ -e^{-0.4t} \cos(0.2t) - e^{-0.4t} \sin(0.2t) + i(e^{-0.4t} \cos(0.2t) - e^{-0.4t} \sin(0.2t)) \end{bmatrix}$$

$$\vec{x}_2 = \text{Re}\{\vec{x}_{cpairs}\} = \begin{bmatrix} e^{-0.4t} \cos(0.2t) \\ -e^{-0.4t} \cos(0.2t) \\ -e^{-0.4t} \cos(0.2t) - e^{-0.4t} \sin(0.2t) \end{bmatrix}$$

$$\vec{x}_3 = \text{Im}\{\vec{x}_{cpairs}\} = \begin{bmatrix} e^{-0.4t} \sin(0.2t) \\ -e^{-0.4t} \sin(0.2t) \\ e^{-0.4t} \cos(0.2t) - e^{-0.4t} \sin(0.2t) \end{bmatrix}$$

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This gives a general solution of

$$\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + c_3 \vec{x}_3(t)$$

$$\vec{x}(t) = c_1 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} e^{-0.4t} \cos(0.2t) \\ -e^{-0.4t} \cos(0.2t) \\ -e^{-0.4t} \cos(0.2t) - e^{-0.4} \sin(0.2t) \end{bmatrix} + c_3 \begin{bmatrix} e^{-0.4t} \sin(0.2t) \\ -e^{-0.4t} \sin(0.2t) \\ e^{-0.4t} \cos(0.2t) - e^{-0.4} \sin(0.2t) \end{bmatrix}$$

Or adding up the vectors we get:

$$\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} = \begin{bmatrix} 2c_1 + c_2 e^{-0.4t} \cos(0.2t) + c_3 e^{-0.4t} \sin(0.2t) \\ c_1 - c_2 e^{-0.4t} \cos(0.2t) - c_3 e^{-0.4t} \sin(0.2t) \\ 2c_1 + c_2(-e^{-0.4t} \cos(0.2t) - e^{-0.4} \sin(0.2t)) + c_3(e^{-0.4t} \cos(0.2t) - e^{-0.4} \sin(0.2t)) \end{bmatrix}$$

Or that

$$x_1(t) = 2c_1 + c_2 e^{-0.4t} \cos(0.2t) + c_3 e^{-0.4t} \sin(0.2t) \quad x_2(t) = c_1 - c_2 e^{-0.4t} \cos(0.2t) - c_3 e^{-0.4t} \sin(0.2t)$$

$$x_3(t) = 2c_1 + c_2(-e^{-0.4t} \cos(0.2t) - e^{-0.4} \sin(0.2t)) + c_3(e^{-0.4t} \cos(0.2t) - e^{-0.4} \sin(0.2t))$$

What happens in the limit that  $t \rightarrow \infty$

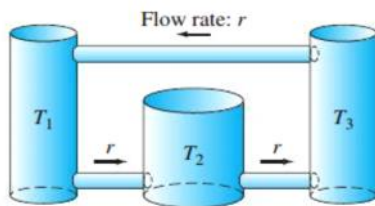
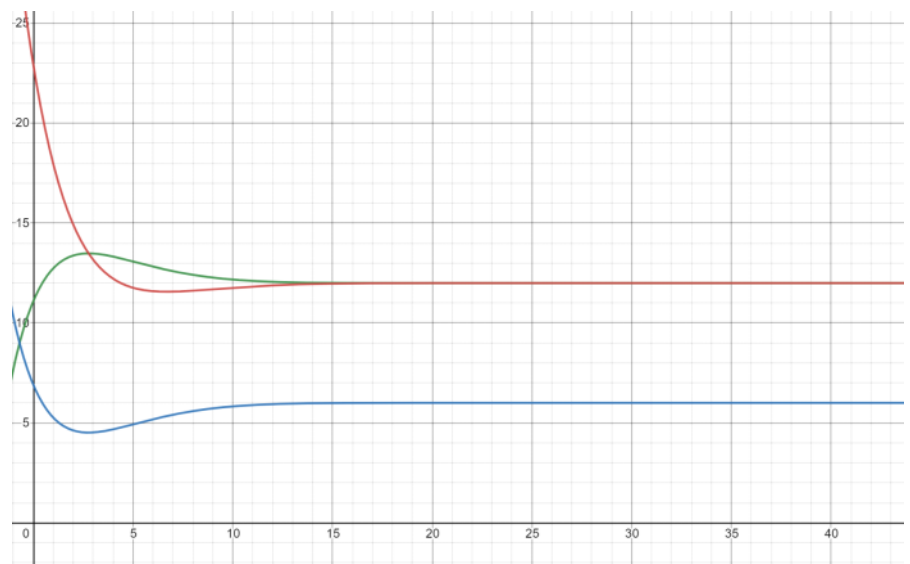


**FIGURE 5.2.5.** The three brine tanks of Example 4.



<https://www.desmos.com/calculator/mdvffqj6am>