

Section 5.3 A Gallery of Solution Curves of a Linear System

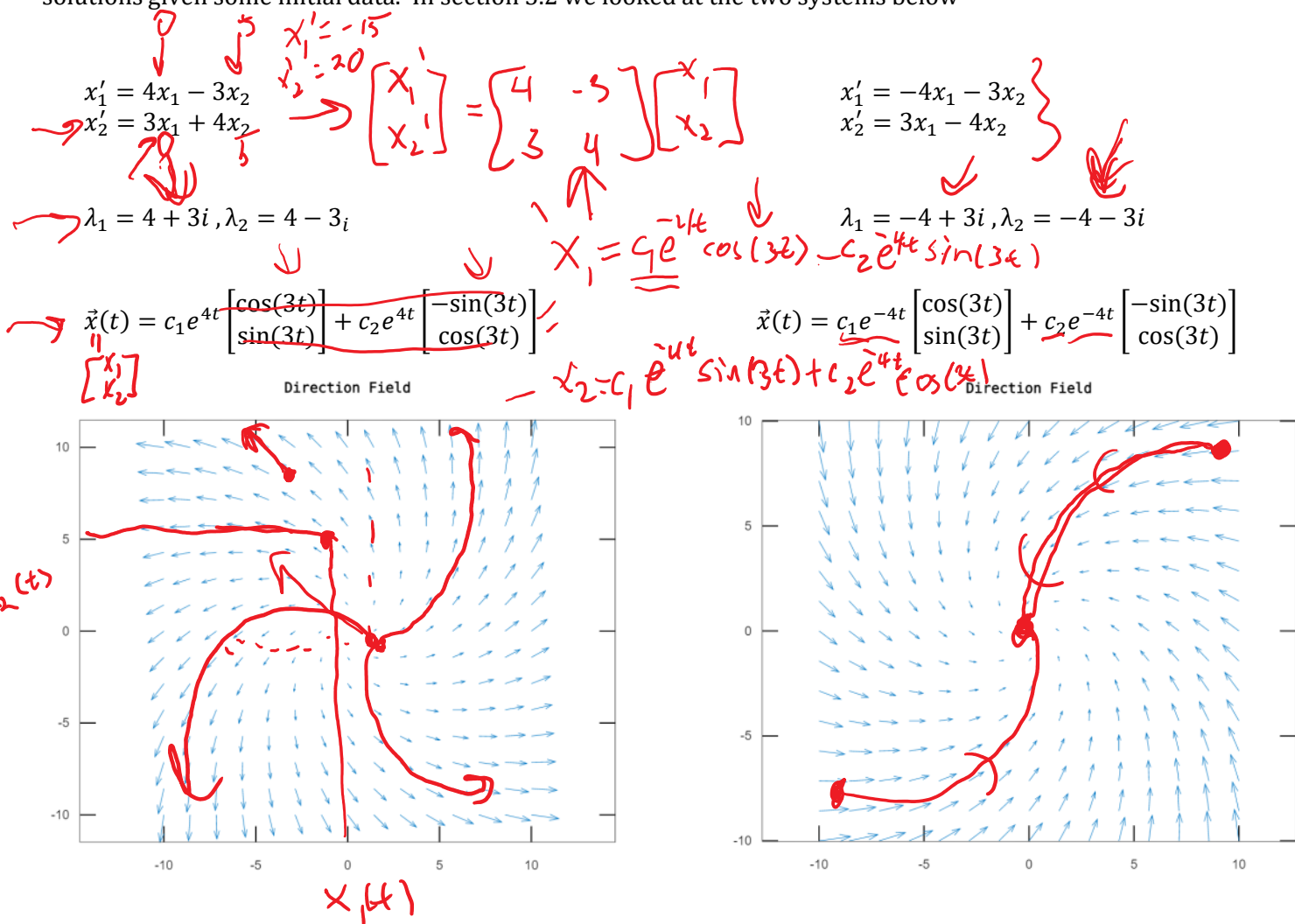
We have been learning to solve systems of first order differential equations of the form:

$$\vec{x}' = A\vec{x}$$

We have also seen that the solutions so systems like this are made up of the eigen vectors and eigen values

$$\vec{x}(t) = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t} + \dots + c_n \vec{v}_n e^{\lambda_n t}$$

So obviously the types of eigen vectors and eigen values we get say an awful lot about the possible solutions given some initial data. In section 5.2 we looked at the two systems below



What we saw was that having complex eigen values the imaginary parts gave us sines and cosines and the real part defined the exponential part. The sines and cosines induce the system to spin and the sign of the real part told us weather the solution was getting bigger or decaying to zero as time went on. As we will see in this section depending on the type of eigen values we get we will immediately be able to say something about the overall behavior of the system. I.E. weather it is spinning, growing, heading to a fixed point etc.

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To help us think what the solutions will do as time goes on it is useful to develop the idea of Euler's method for a system of n equations. In 1-d we had the following formulation.

Given the initial value problem:

$$\frac{dy}{dx} = f(x, y),$$

$$y(x_0) = y_0$$

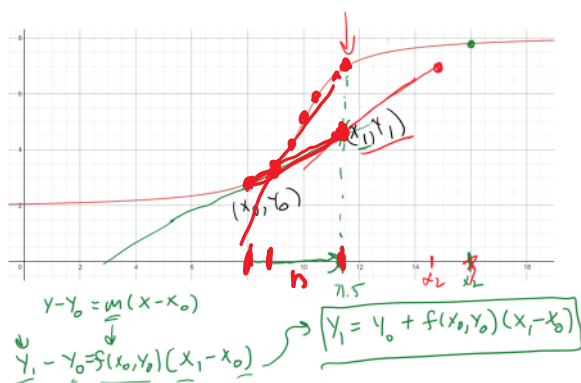
$$y_1 = y_0 + h \cdot f(x_0, y_0)$$

Euler's method with step size h consists of applying the iterative formula below with $n \geq 0$.

$$y_{n+1} = y_n + h \cdot f(x_n, y_n)$$

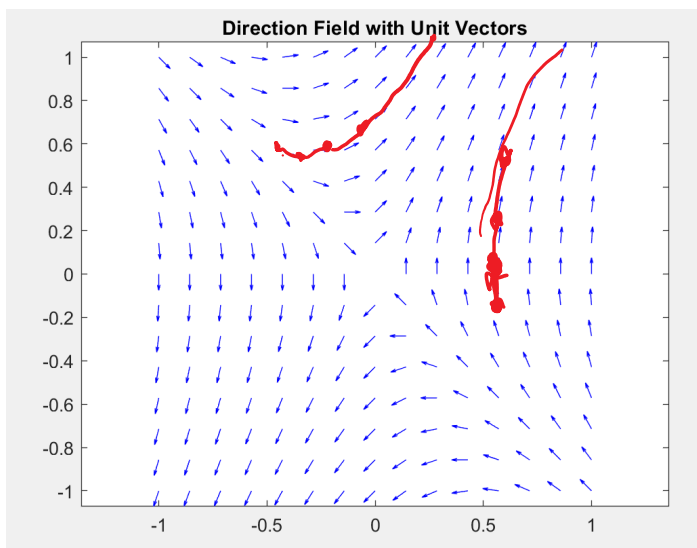
To calculate successive approximations y_1, y_2, y_3 to the true values $y(x_1), y(x_2), y(x_3)$ of the function $y(x)$ that is a solution to the initial value problem at the points x_1, x_2, x_3 .

Essentially what we did was start at the point y_0 and then follow the derivative (tangent line) to the next point of the solution y_1



We found that smaller step sizes usually produced better approximations of the solution.

We can play a very similar game here since $\vec{x}' = A\vec{x}$ we can try to go from a point $\vec{x}^0 = \begin{bmatrix} x_1^0 \\ x_2^0 \end{bmatrix}$ to a point $\vec{x}^1 = \begin{bmatrix} x_1^1 \\ x_2^1 \end{bmatrix}$ by following the vector $\vec{x}' = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix}$ from the first point to the next point.



$$\vec{x}^1 = \vec{x}^0 + h\vec{x}'$$

But $\vec{x}' = A\vec{x}$

$$\begin{aligned} \vec{x}^1 &= \vec{x}^0 + hA\vec{x}^0 \\ \vec{x}^2 &= \vec{x}^1 + hA\vec{x}^1 \\ &\vdots \\ \vec{x}^n &= \vec{x}^{n-1} + hA\vec{x}^{n-1} \end{aligned}$$

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Euler 2-d

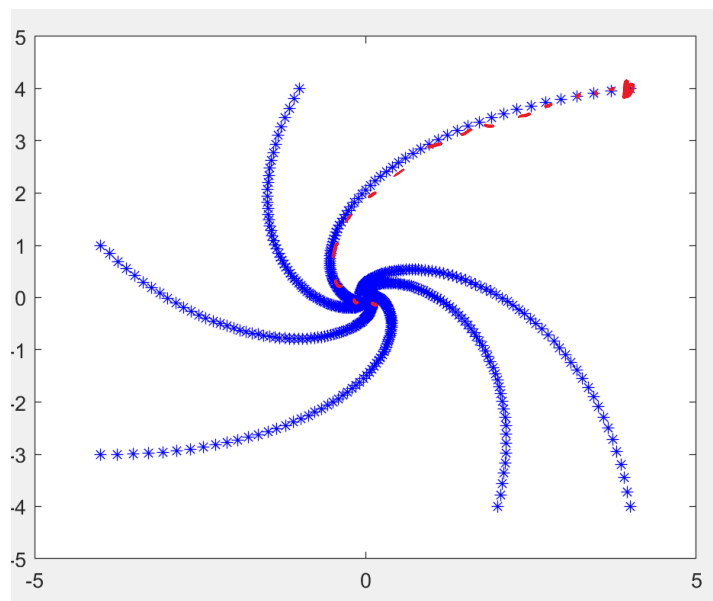
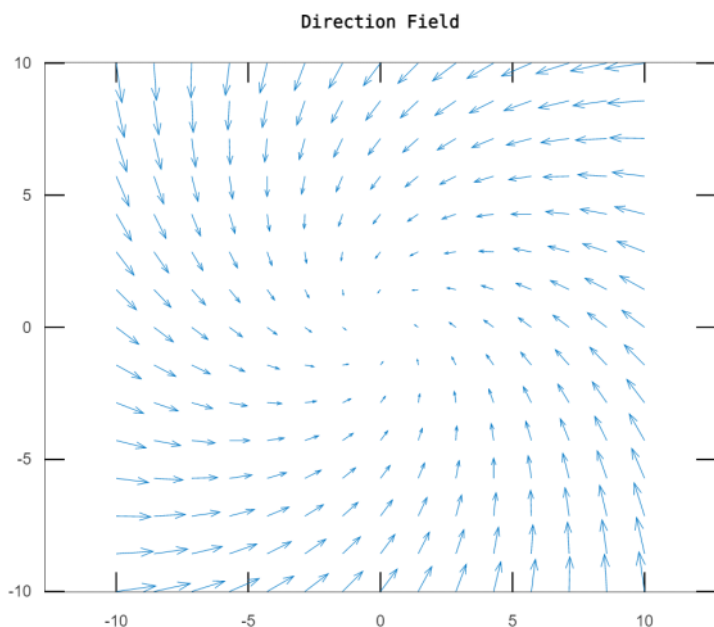
For a system $\vec{x}' = A\vec{x}$ in 2d we can find the euler approximates with the following formula

$$\begin{bmatrix} x_1^n \\ x_2^n \end{bmatrix} = \begin{bmatrix} x_1^{n-1} \\ x_2^{n-1} \end{bmatrix} + hA \begin{bmatrix} x_1^{n-1} \\ x_2^{n-1} \end{bmatrix}$$

Where h is our chosen step size

Some code you could use in octave or matlab to explore 2d systems.

```
A=[-4,-3;3,-4]; %coefficient matrix
for x0=[-4;1],[4;4],[2;-4],[-4;-3],[4;-4],[-1;4] %initl conditions to try loop
x=x0; %set initial condition
h=0.01; %step size
for i=1:100 %loop for euler
plot(x(1),x(2),'*','Color','blue') % plot starting point and all other points
xlim([-5,5]) %sets window on plot
ylim([-5,5])
hold on % allows for next point to be overlayed over previous plot
pause(0.01) % pauses between plots to make it look like a movie
x=A*x*h+x; % step to next point go back up and plot
end
end
```



Our system $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -4 & -3 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

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Let's first deal with the types of overall behavior we can expect based on different combinations of real eigen values. For the most part we will examine 2-d systems since they are easy to understand and plot.

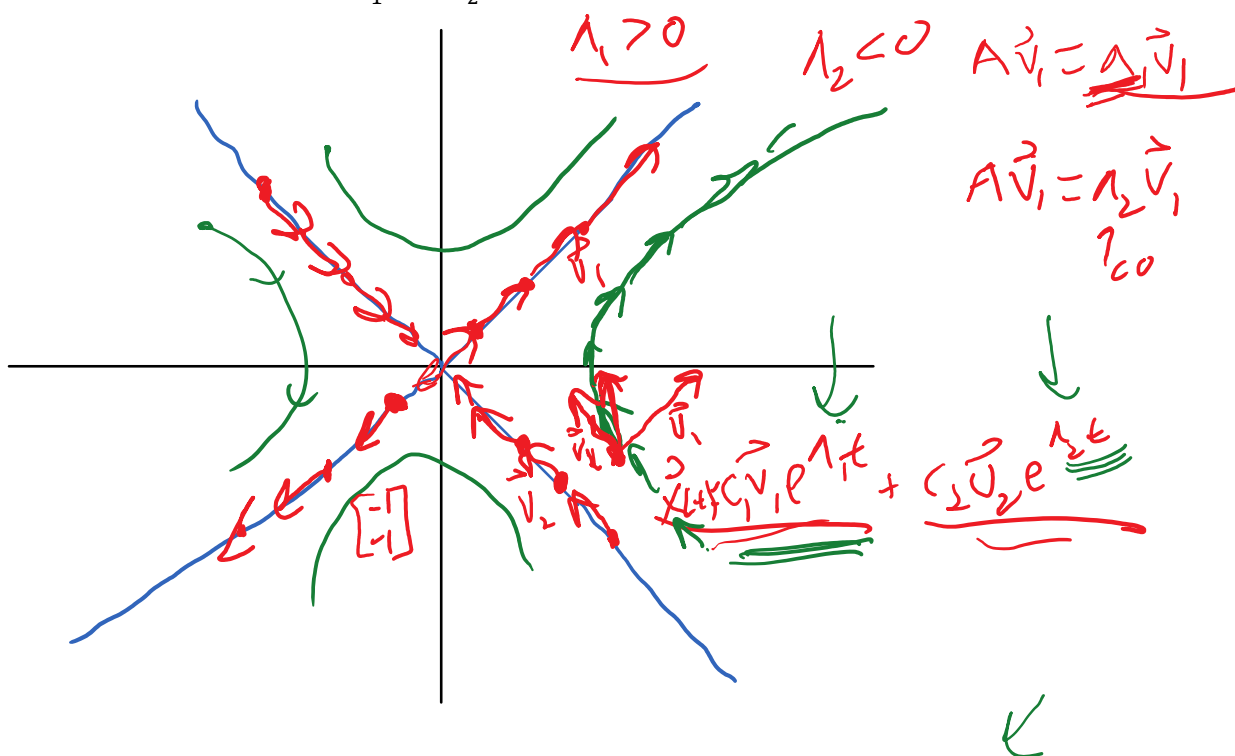
Real Eigen Values

In this case our solution would look like:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t}$$

$$c_1 A \vec{v}_1 e^{\lambda_1 t} = c_1 \lambda_1 \vec{v}_1 e^{\lambda_1 t}$$

To analyze how this kind of solution will behave in general let's first think about what might happen if we have our initial condition on one of the eigen vectors. One fact we will need is that with distinct eigen values their eigen vectors will be linearly independent. Which means that if we have initial data on one of them the constant multiplying the other will have to be zero. So if we start on $\vec{v}_1$ then c_2 will have to be zero.



Suppose we have an initial condition on $\vec{v}_1$. If we apply A to $\vec{x}$ we would have $Ac_1 \vec{v}_1 e^0 = c_1 A \vec{v}_1 = c_1 \lambda_1 \vec{v}_1$ so the derivative would tell us to just continue along $\vec{v}_1$ in the direction defined by the sign of λ_1 the same is true for an initial condition chosen to initially lie on $\vec{v}_2$, $Ac_2 \vec{v}_2 e^0 = c_2 A \vec{v}_2 = c_2 \lambda_2 \vec{v}_2$ let's draw this above.

If we started somewhere not on either we would have a combination of the two

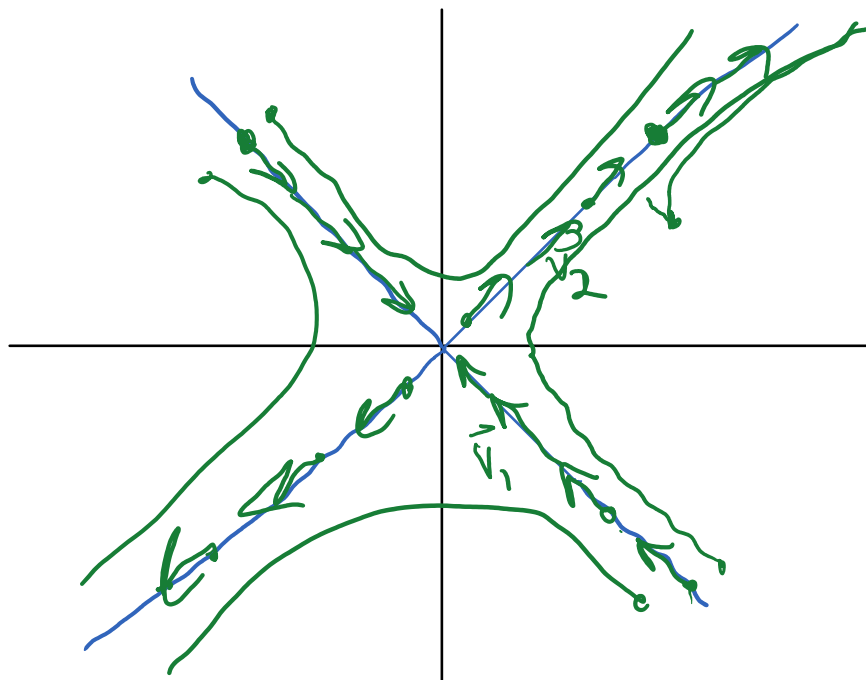
$$A(c_1 \vec{v}_1 e^0 + c_2 \vec{v}_2 e^0) = Ac_1 \vec{v}_1 e^0 + Ac_2 \vec{v}_2 e^0 = c_1 \lambda_1 \vec{v}_1 + c_2 \lambda_2 \vec{v}_2$$

What might that look like above?

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Case: $\lambda_1 \neq \lambda_2$, $\lambda_1 < 0 < \lambda_2$ (not equal and of opposite sign)

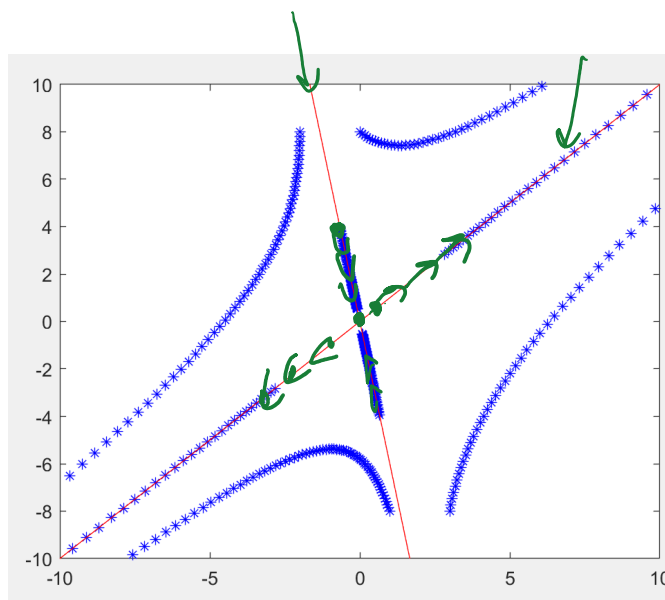
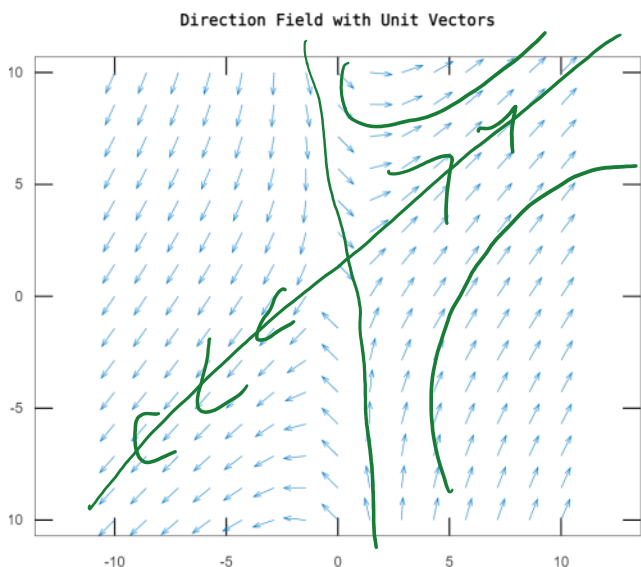
$$c_1 e^{\lambda_1 t} \vec{v}_1 + c_2 e^{\lambda_2 t} \vec{v}_2$$



Let's consider the system:
$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 6 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Which has eigen values and eigen vectors

$$\lambda_1 = -2, \vec{v}_1 = \begin{bmatrix} -1 \\ 6 \end{bmatrix}, \lambda_2 = 5, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



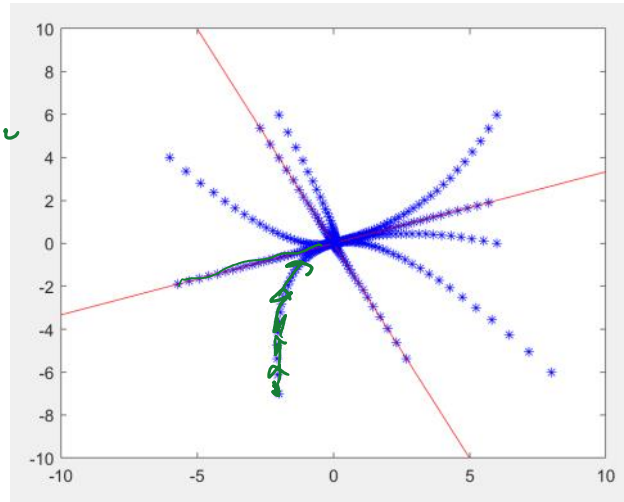
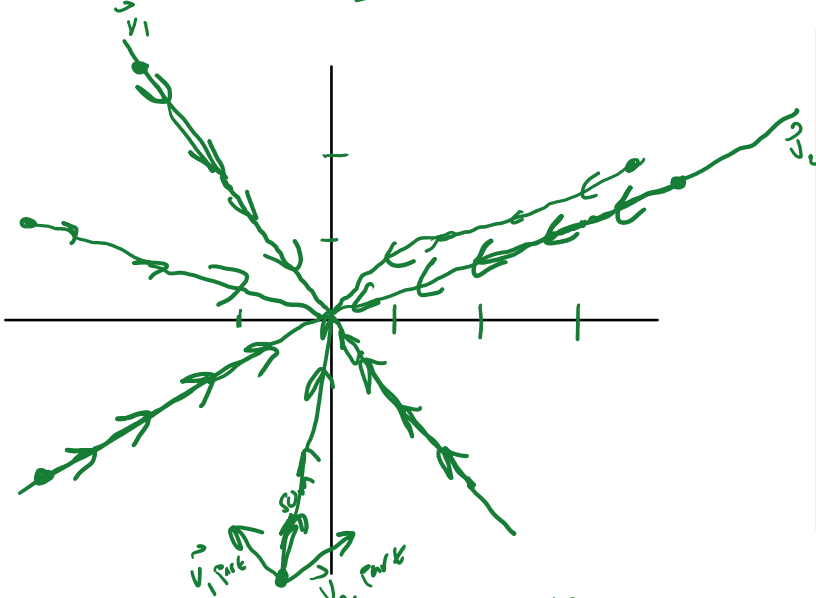
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Case: $\lambda_1 \neq \lambda_2$, $0 < \lambda_1 < \lambda_2$ (both ^{negative} ~~positive~~)

Let's consider the system: $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -8 & 3 \\ 2 & -13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\lambda_1 = -14, \vec{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \lambda_2 = -7, \vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\vec{x}(t) = c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{-14t} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{-7t}$

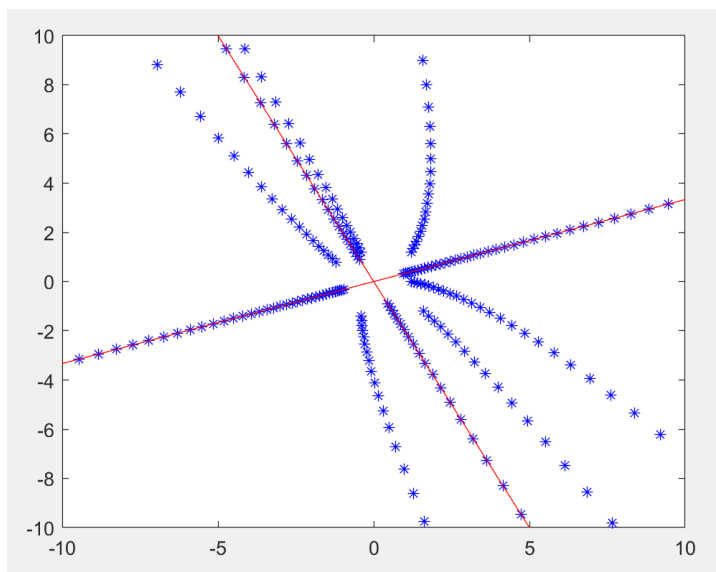
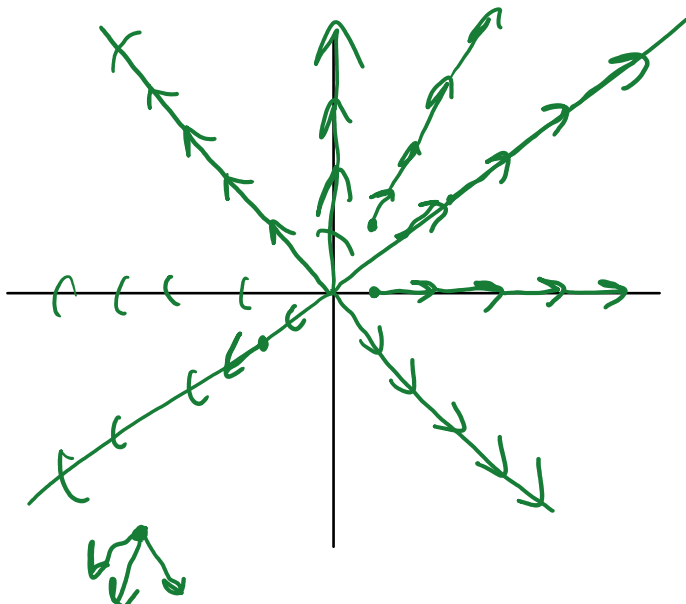


Case: $\lambda_1 \neq \lambda_2$, $\lambda_1 < \lambda_2 < 0$ (both ^{positive} ~~negative~~)

Let's consider the system: $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 8 & -3 \\ -2 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\lambda_1 = 14, \vec{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \lambda_2 = 7, \vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$

$\vec{x}(t) = c_1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} e^{14t} + c_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix} e^{7t}$



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Principle of time reversal

Let $x(t)$ be a solution of the 2-d linear system

$$x(t) = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t}$$

$$x(-t) = -\frac{1}{c_1} \vec{v}_1 e^{\lambda_1 t} - \frac{1}{c_2} \vec{v}_2 e^{\lambda_2 t}$$

$$\vec{x}' = A\vec{x}$$

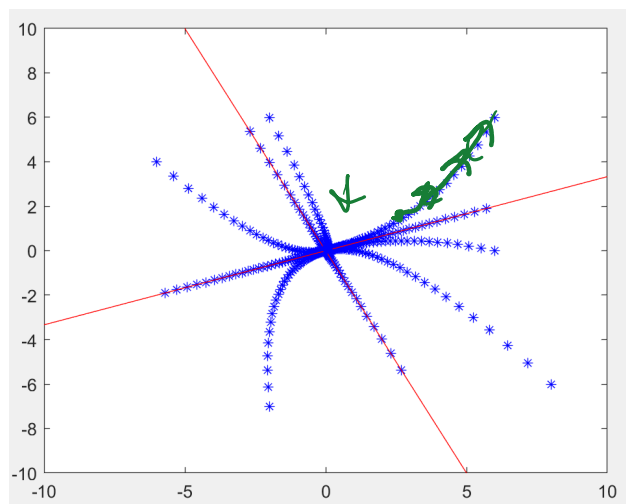
Then the function $\vec{x}(-t)$ is the solution of the system

$$\vec{x}' = -A\vec{x}$$

Let's consider the system: $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -8 & 3 \\ 2 & -13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

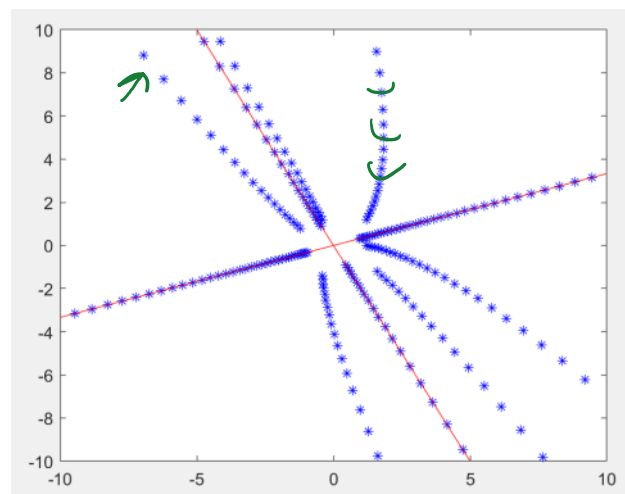
$$\lambda_1 = -14, \vec{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \lambda_2 = -7, \vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$c_1 e^{-14t} \vec{v}_1 + c_2 e^{-7t} \vec{v}_2$$



Let's consider the system: $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 8 & -3 \\ -2 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

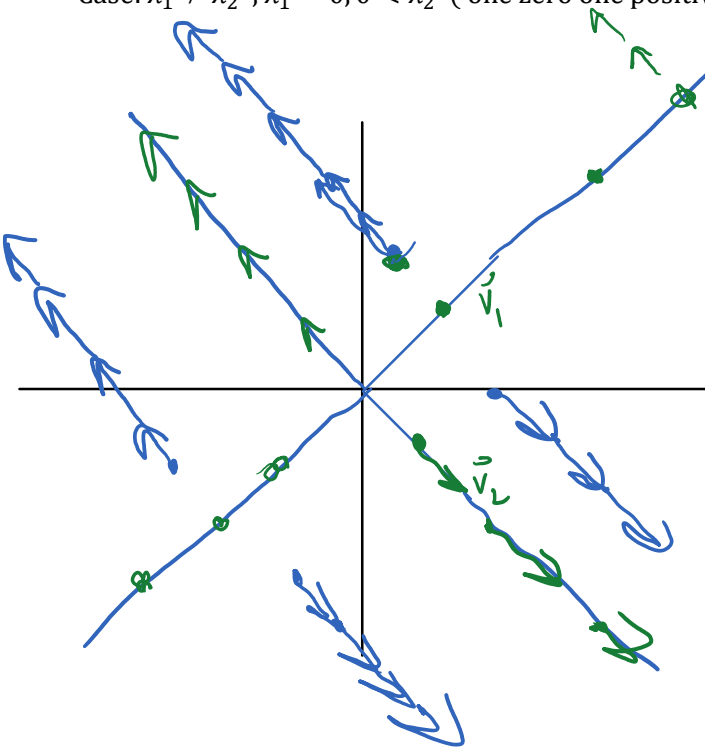
$$\lambda_1 = 14, \vec{v}_1 = \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \lambda_2 = 7, \vec{v}_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$



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$$\vec{x}(0) = \vec{v}_1 = c_1 \vec{v}_1 e^{0t} + c_2 \vec{v}_2 e^{\lambda_2 t}$$

Case: $\lambda_1 \neq \lambda_2$, $\lambda_1 = 0$, $0 < \lambda_2$ (one zero one positive)



What will the solution look like?

$$\vec{x}(t) = c_1 \vec{v}_1 e^{0t} + c_2 \vec{v}_2 e^{\lambda_2 t}$$

$$= c_1 \vec{v}_1 + c_2 \vec{v}_2 e^{\lambda_2 t}$$

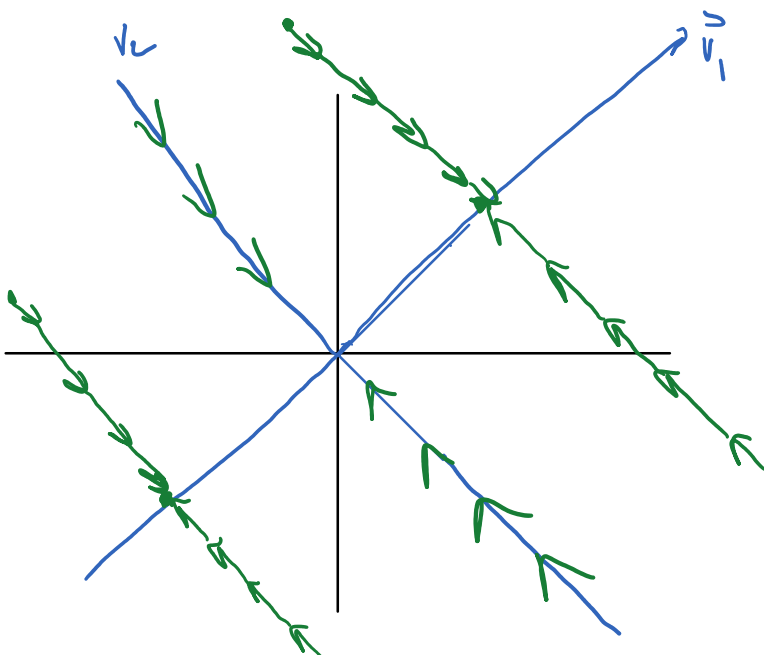
Start on $\vec{v}_1$ $c_2 = 0$

$$\vec{x}(t) = c_1 \vec{v}_1$$

$$\vec{x}'(t) = 0$$

$$\vec{x}'(t) = \lambda_2 c_2 \vec{v}_2 e^{\lambda_2 t}$$

Case: $\lambda_1 \neq \lambda_2$, $\lambda_1 = 0$, $0 < \lambda_2$ (one zero one negative)



What will the solution look like?

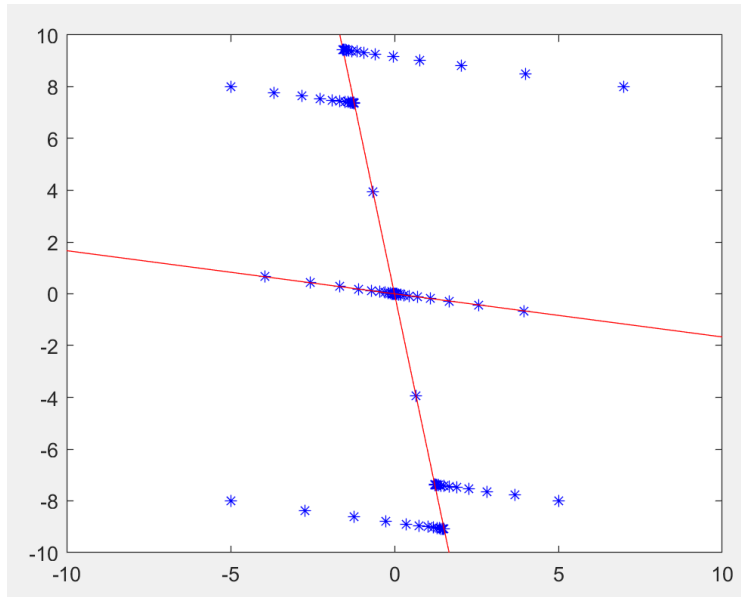
$$\vec{x}(t) = c_1 \vec{v}_1 + c_2 \vec{v}_2 e^{-|\lambda_2| t}$$

$$\vec{x}'(t) = \lambda_2 c_2 \vec{v}_2 e^{\lambda_2 t} \quad \lambda_2 < 0$$

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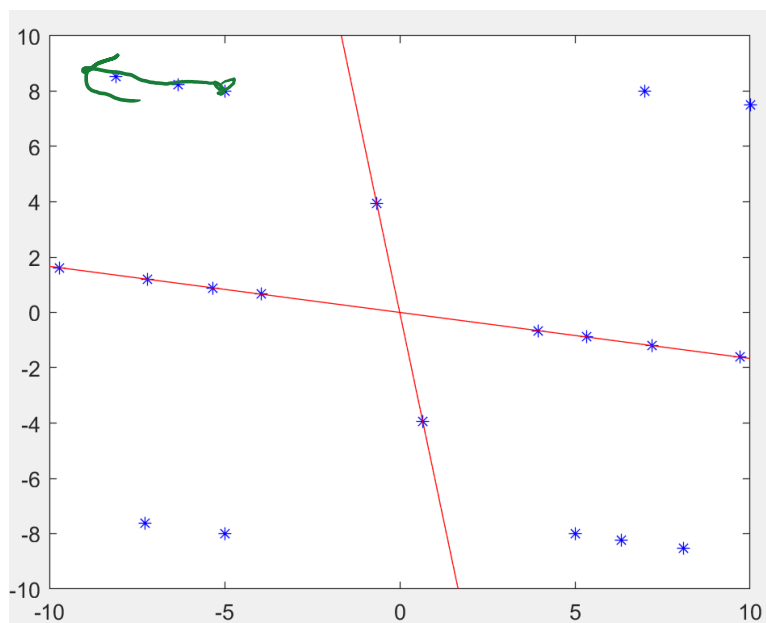
Let's consider the system: $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -36 & -6 \\ 6 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\lambda_1 = -35, \vec{v}_1 = \begin{bmatrix} 6 \\ -1 \end{bmatrix}, \lambda_2 = 0, \vec{v}_2 = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$



Let's consider the system: $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 36 & 6 \\ -6 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\lambda_1 = 35, \vec{v}_1 = \begin{bmatrix} 6 \\ -1 \end{bmatrix}, \lambda_2 = 0, \vec{v}_2 = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$



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This covers all of the possibilities when we have real and distinct eigen values, but what if we have a repeated eigen value?

There are two possibilities with a repeated eigen value, we can have two linearly independent eigen vectors or two linearly dependent eigenvectors.

Let's suppose that our coefficient matrix A has a repeated eigen value with linearly independent eigen vectors.

This means that:

$$A\vec{v}_1 = \lambda\vec{v}_1$$

$$A\vec{v}_2 = \lambda\vec{v}_2$$

For the same lambda.

$$\begin{bmatrix} 10 \\ 5 \end{bmatrix} = 10 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

There is a fact from linear algebra that if two 2-d vectors are linearly independent then for any other 2d vector $\vec{c}$ I can always find c_1, c_2 such that

$$\vec{c} = c_1\vec{v}_1 + c_2\vec{v}_2$$

Well this means that

$$A\vec{c} = A(c_1\vec{v}_1 + c_2\vec{v}_2) = c_1A\vec{v}_1 + c_2A\vec{v}_2 = c_1\lambda\vec{v}_1 + c_2\lambda\vec{v}_2 = \lambda(c_1\vec{v}_1 + c_2\vec{v}_2) = \lambda\vec{c}$$

This means that if A has a repeated eigen value with two linearly independent eigen vectors then every other vector is also an eigen vector.

Incidentally this means A must be of the form

$$A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}$$

Further our solution would be of the form

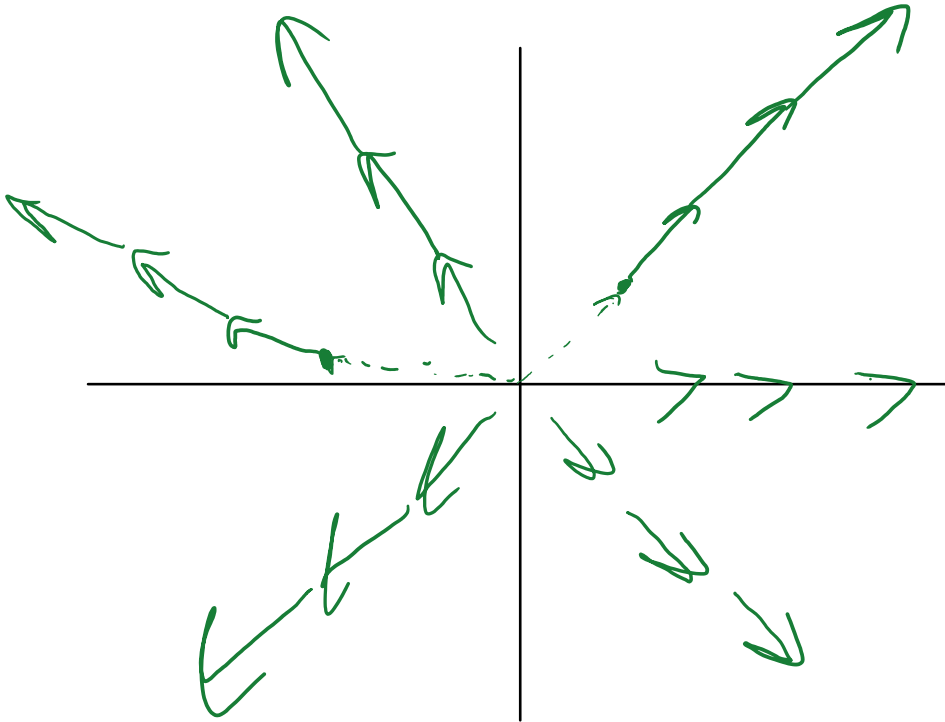
$$\vec{x}(t) = (c_1\vec{v}_1e^{\lambda t} + c_2\vec{v}_2e^{\lambda t}) = (c_1\vec{v}_1 + c_2\vec{v}_2)e^{\lambda t} = \vec{c}e^{\lambda t} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{\lambda t}$$

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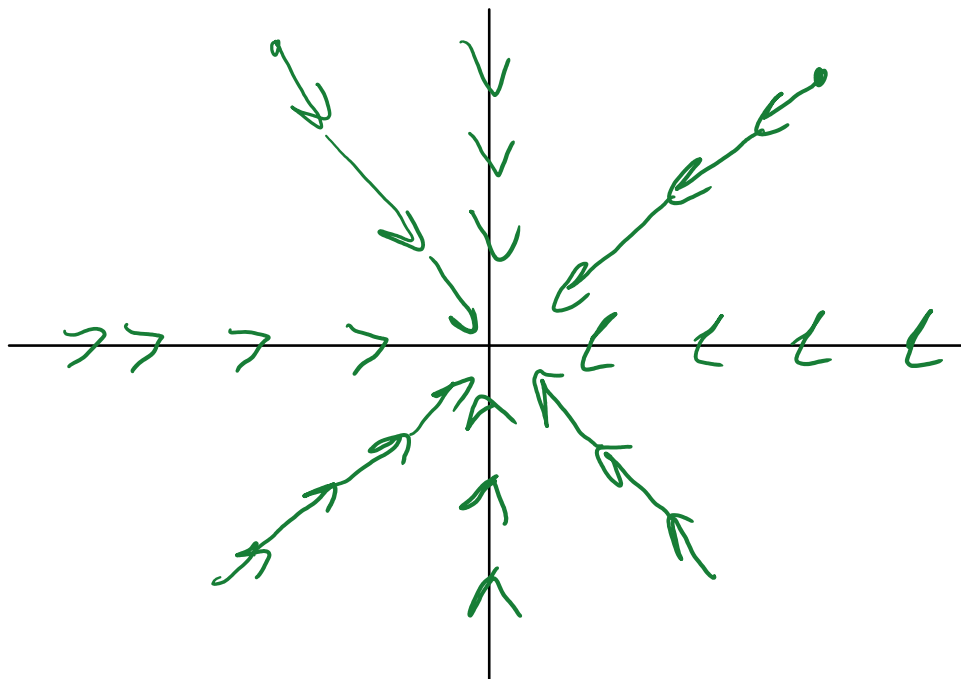
So the solution looks like $\vec{x}(t) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} e^{\lambda t}$

We call the following situations a nodal source or sink

What if $\lambda > 0$



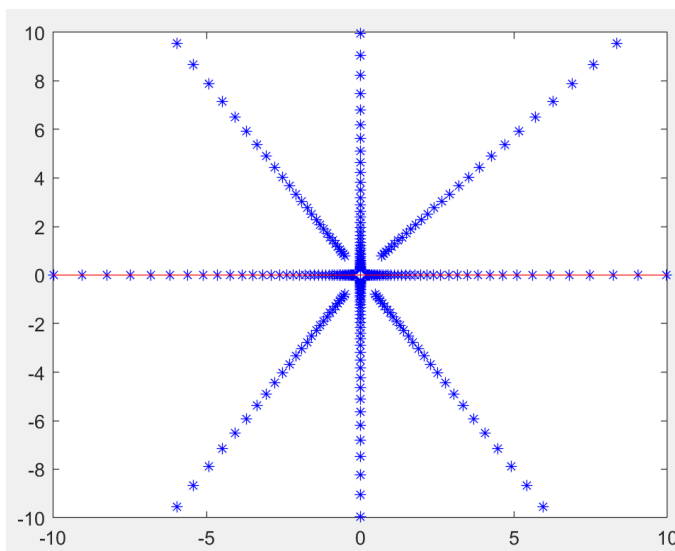
What if $\lambda < 0$



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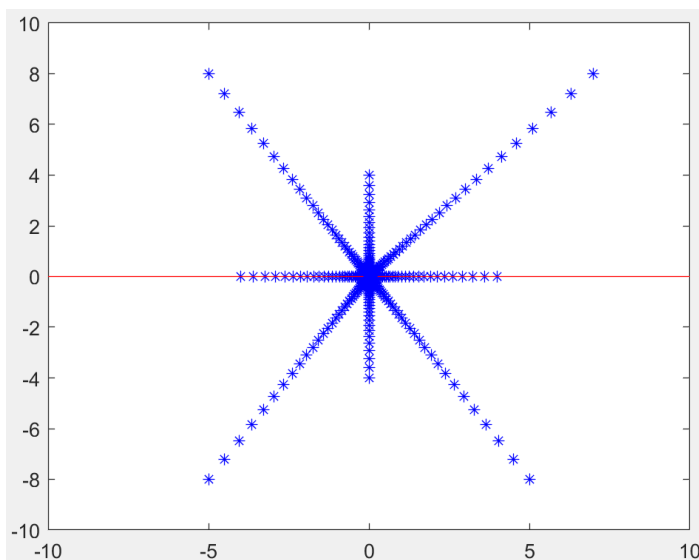
Let's consider the system: $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\lambda_1 = 2, \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \lambda_2 = 2, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$



Let's consider the system: $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\lambda_1 = -2, \vec{v}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \lambda_2 = -2, \vec{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

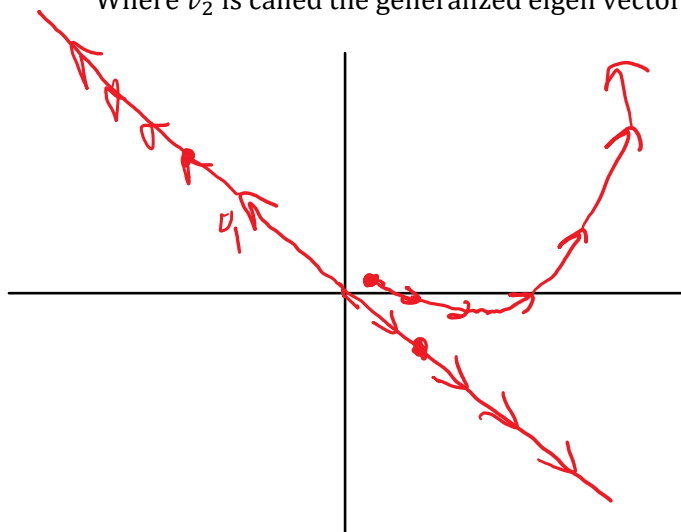


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In the case that we do not get two linearly independent eigen vectors from our repeated eigen value the solution to the system $\vec{x}' = A\vec{x}$ is of the form

$$\vec{x}(t) = c_1 \vec{v}_1 e^{\lambda t} + c_2 (\vec{v}_1 t + \vec{v}_2) e^{\lambda t}$$

Where $\vec{v}_2$ is called the generalized eigen vector which we will discuss later.



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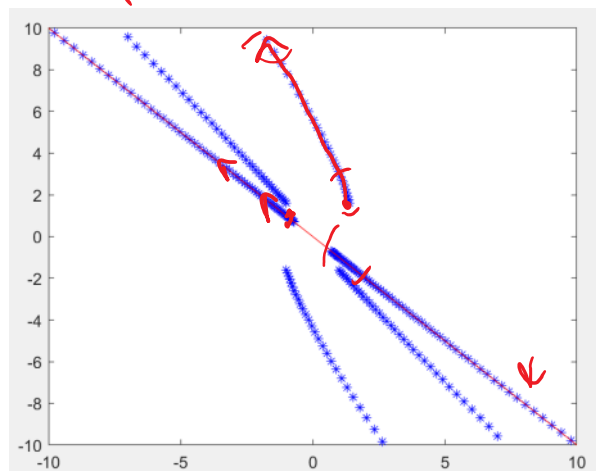
What happens if we start on $\vec{v}_1$

What if off of it?

Let's consider the system: $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

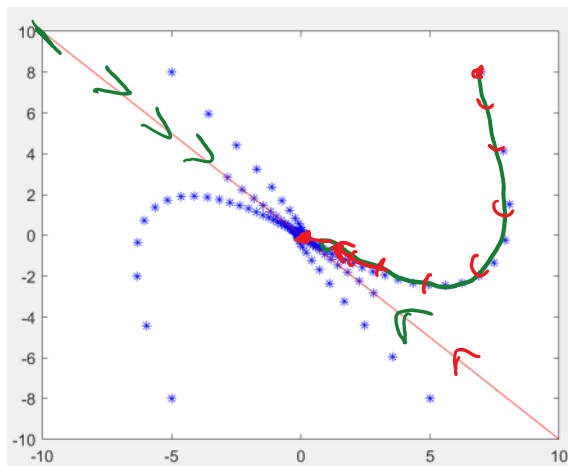
$$\lambda_1 = 4, \vec{v}_1 = \begin{bmatrix} -3 \\ 3 \end{bmatrix}, \lambda_2 = 4, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$(A - \lambda I) \vec{v} = 0$$



Let's consider the system: $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = -\begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\lambda_1 = -4, \vec{v}_1 = \begin{bmatrix} -3 \\ 3 \end{bmatrix}, \lambda_2 = -4, \vec{v}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



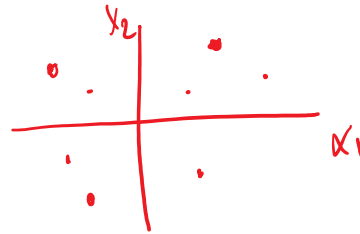
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What if the repeated eigen value is $\lambda = 0$

Well if we have two linearly independent eigen vectors this means $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

In which case $\vec{x}' = A\vec{x} = 0$ which means that the vector $\vec{x}$ had to be constant!

$$\vec{x}(t) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$



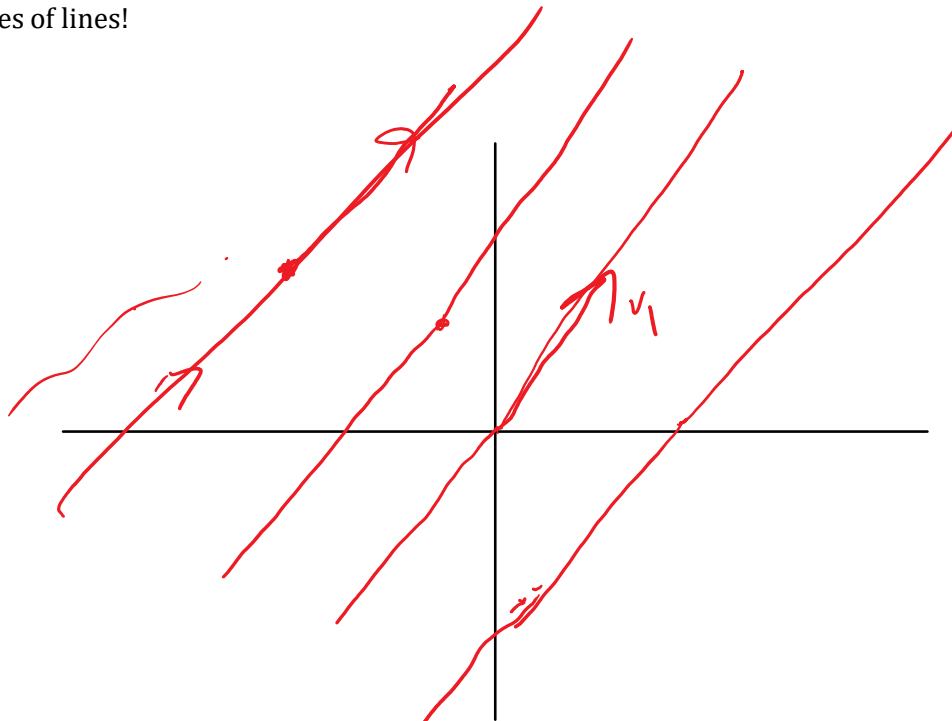
If not then we again have a solution of the form

$$\vec{x}(t) = c_1 \vec{v}_1 e^{\lambda t} + c_2 (\vec{v}_1 t + \vec{v}_2) e^{\lambda t}$$

But with $\lambda = 0$ we just have

$$\vec{x}(t) = c_1 \vec{v}_1 + c_2 (\vec{v}_1 t + \vec{v}_2) = (c_1 \vec{v}_1 + c_2 \vec{v}_2) + c_2 \vec{v}_1 t$$

Or just a series of lines!



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Complex Eigen Values

Purely imaginary $\leftarrow a+bi$ $a=0$ $\lambda_1 = 0+bi$
 $\lambda_2 = 0-bi$

We saw before when solving systems using the eigen value method that when we have complex eigen values the solution of the system takes the form:

$$\vec{x}(t) = e^{at} \left[A \left(\vec{a} \cos(bt) - \vec{b} \sin(bt) \right) + B \left(\vec{b} \cos(bt) + \vec{a} \sin(bt) \right) \right]$$

Where $\lambda_1 = a + bi$ and $\lambda_2 = a - bi$ however if we have purely imaginary eigen values this means $a = 0$ and our solution becomes:

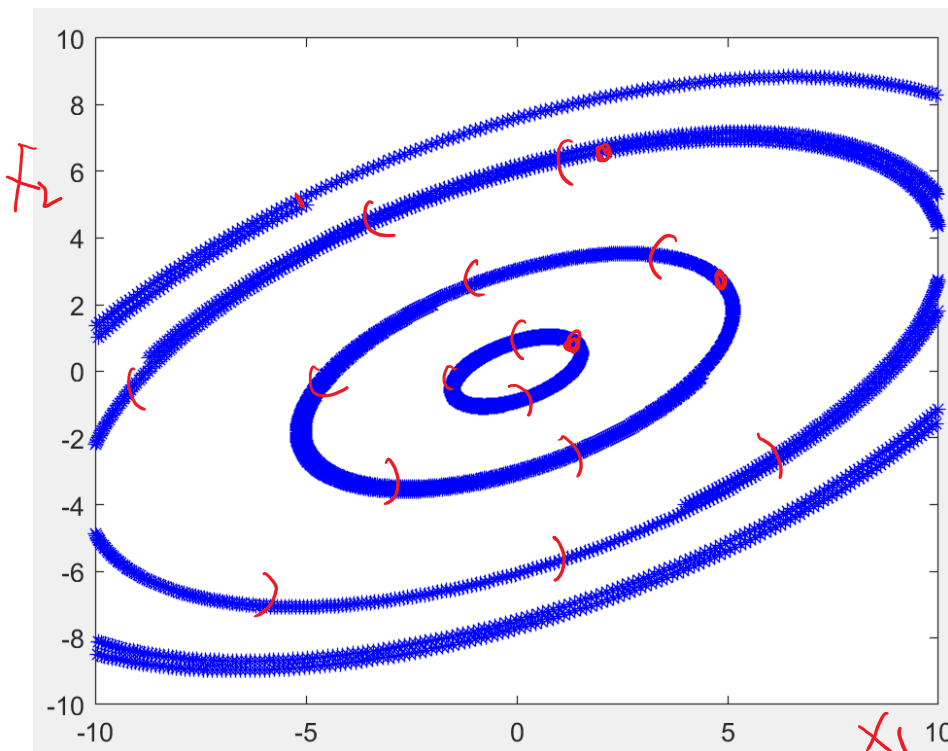
$$\vec{x}(t) = \left[A \left(\vec{a} \cos(bt) - \vec{b} \sin(bt) \right) + B \left(\vec{b} \cos(bt) + \vec{a} \sin(bt) \right) \right] = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

When separating the real and imaginary parts we say that the vector $\vec{x}$ is made up of the two real solutions.

$$\begin{aligned} \vec{x}_1 &= \left(\vec{a} \cos(bt) - \vec{b} \sin(bt) \right) \\ \vec{x}_2 &= \left(\vec{b} \cos(bt) + \vec{a} \sin(bt) \right) \end{aligned}$$

Which just defines an ellipse!

Let's consider the system: $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 6 & -17 \\ 8 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$
 $\lambda_1 = 10i, \vec{v}_1 = \begin{bmatrix} 3+5i \\ 4 \end{bmatrix}, \lambda_2 = -10i, \vec{v}_2 = \begin{bmatrix} 3-5i \\ 4 \end{bmatrix}$



Section 5.3 A Gallery of Solution Curves of a Linear System

Complex eigen values give us spirals! When we don't have a non-zero real part our solution is of the form

$$\begin{cases} \vec{x}_1 = e^{at} (\vec{a} \cos(bt) - \vec{b} \sin(bt)) \\ \vec{x}_2 = e^{at} (\vec{b} \cos(bt) + \vec{a} \sin(bt)) \end{cases}$$

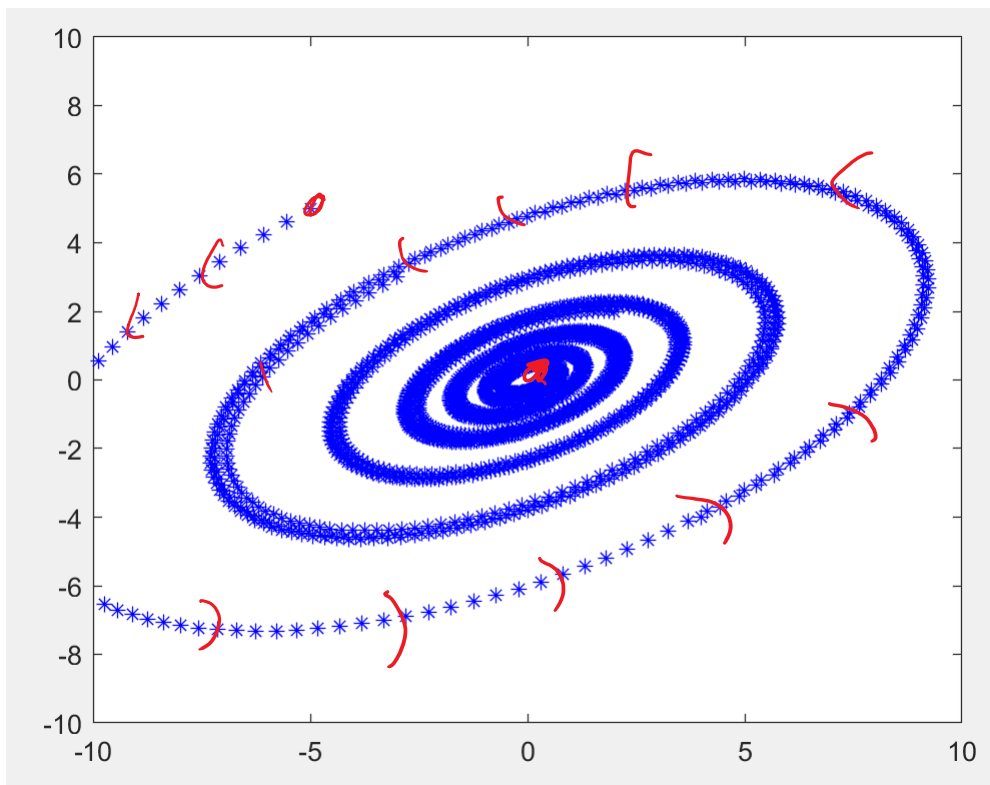
If a is positive the values will grow exponentially if $a < 0$ they will decay to zero!

Let's consider the system: $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 5 & -17 \\ 8 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$$\lambda_1 = -1 + 10i, \vec{v}_1 = \begin{bmatrix} 3 + 5i \\ 4 \end{bmatrix}, \lambda_2 = -1 - 10i, \vec{v}_2 = \begin{bmatrix} 3 - 5i \\ 4 \end{bmatrix}$$

Our solution has the form:

$$\vec{x}(t) = e^{at} \left[A (\vec{a} \cos(bt) - \vec{b} \sin(bt)) + B (\vec{b} \cos(bt) + \vec{a} \sin(bt)) \right]$$



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Of course if we have a positive real part of the eigen value things will spiral out ward!

Let's consider the system: $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} -5 & 17 \\ -8 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\lambda_1 = 1 + 10i, \vec{v}_1 = \begin{bmatrix} 3 + 5i \\ 4 \end{bmatrix}, \lambda_2 = 1 - 10i, \vec{v}_2 = \begin{bmatrix} 3 - 5i \\ 4 \end{bmatrix}$

