

Section 5.5 Multiple Eigen Value Solutions

In Section 5.3 we stated that when we have a repeated eigen value of a matrix there may or may not be more than one eigen vector associated with it.

Let's consider a system of two linear first order equations of the form $\vec{x}' = A\vec{x}$

And let's suppose that we have a repeated eigen value of λ that only has one eigen vector associated with it $\vec{v}_1$

We said that in this case the solution to the system is $\vec{x} = c_1 \vec{v}_1 e^{\lambda t} + c_2 (\vec{v}_1 t + \vec{v}_2) e^{\lambda t}$

Where $\vec{v}_2$ is called a generalized eigen vector. Let's see how we might find this eigen vector by plugging it in to the differential equation itself

So for the $\vec{x}(t)$ above to be a solution the generalized eigen vector $\vec{v}_2$ must satisfy $(A - \lambda I)\vec{v}_2 = \vec{v}_1$

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Let's consider the system: $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = -\begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and find the general solution and the generalized eigen vector.

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Let's just consider the general case of a repeated eigen value for a matrix A . We know that the way we find eigen values is to solve the characteristic polynomial coming from the equation

$$\det(A - \lambda I) = 0$$

For an $n \times n$ matrix A we know we could factor that equation into a product of linear terms involving the eigen values.

$$(\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n) = 0$$

If we have a repeated eigen value this might look more like

$$(\lambda - \lambda_1)^m (\lambda - \lambda_2) \cdots (\lambda - \lambda_{n-k}) = 0$$

Here we would say that the eigen value λ_1 would have algebraic multiplicity m . If when solving for the eigen vectors for λ_1 we find m linearly independent ones we say that λ_1 is **complete**. If there are only $p < m$ linearly independent eigen vectors we call it a **defective** eigen value. Further we call the number of missing eigen vectors $d = m - p$ the **defect**. This will also be the number of generalized eigen vectors that we can find.

We can find all of the generalized eigen vectors belonging to a defective eigen value successively. Suppose that $\vec{v}_1$ is one of the regular eigen vectors of λ, A which means that it satisfies.

$$(A - \lambda I)\vec{v}_1 = 0$$

From what we saw before a generalized eigen vector can be found solving

$$(A - \lambda I)\vec{v}_2 = \vec{v}_1$$

for $\vec{v}_2$. The next generalized eigen vector may be found with

$$(A - \lambda I)\vec{v}_3 = \vec{v}_2$$

All the way up to how ever many generalized eigen vectors might be generated from $\vec{v}_1$

$$\vdots \\ (A - \lambda I)\vec{v}_k = \vec{v}_{k-1}$$

Note the following pattern using the above sequence:

$$(A - \lambda I)(A - \lambda I)\vec{v}_2 = (A - \lambda I)\vec{v}_1 = 0 \rightarrow (A - \lambda I)^2 \vec{v}_2 = 0$$

then

$$(A - \lambda I)^2 (A - \lambda I)\vec{v}_3 = (A - \lambda I)^2 \vec{v}_2 = 0 \rightarrow (A - \lambda I)^3 \vec{v}_3 = 0$$

And so on until

$$(A - \lambda I)^k \vec{v}_k = 0$$

Here we would say that k is the rank of the generalized eigen vector.

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We would call the set of vectors we found $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ a length k chain of generalized eigen vectors based on the eigen vector $\vec{v}_1$.

It may not be that we get all of our generalized eigen vectors from $\vec{v}_1$ however. If we have other regular eigen vectors coming from lambda they may also be generators of some of the generalized eigen vectors of λ and A .

Let's choose a different notation for generalized eigen vectors now, let's denote them with a u and the regular eigen vectors with a v .

To illustrate the possibilities with a defective eigen value and its generalized eigen vectors let's suppose the matrix A has an eigen value λ with multiplicity $m = 4$ and defect $d = \dots$.

$d = 0$, Then we have $\{\vec{v}_1\}, \{\vec{v}_2\}, \{\vec{v}_3\}, \{\vec{v}_4\}$ four length 1 chains made up of regular eigen vectors

$d = 1$, Then we have $\{\vec{v}_1, \vec{u}_1\}, \{\vec{v}_2\}, \{\vec{v}_3\}$ two length 1 chains of regular eigen vectors and a length 2 chain of a regular eigen vector and a generalized eigen vector coming from it.

$d = 2$ Then we can have $\{\vec{v}_1, \vec{u}_1\}, \{\vec{v}_2, \vec{u}_2\}$ two length two chains with a regular eigen vector and a generalized eigen vector generated from each.

$d = 2$ We could also have $\{\vec{v}_1, \vec{u}_1, \vec{u}_2\}, \{\vec{v}_2\}$ One length 3 chain and one length 1 chain.

$d = 3$ $\{\vec{v}_1, \vec{u}_1, \vec{u}_2, \vec{u}_3\}$ a length 4 chain with generalized eigen vectors all coming from the only regular one found.

Suppose we indeed have an eigen value λ with multiplicity 4 and defect 2, how can we know weather the vectors we need come in the form $\{\vec{v}_1, \vec{u}_1\}, \{\vec{v}_2, \vec{u}_2\}$ or $\{\vec{v}_1, \vec{u}_1, \vec{u}_2\}, \{\vec{v}_2\}$ Well we will be able to know but first let's give an aloorithim for constructing the chains...

We notice that in every case above the length of the chain was at most $d + 1$ so we could begin looking for generalized eigen vectors by solving

$$(A - \lambda I)^{d+1} \vec{u}_{d+1} = 0$$

If we get a solution call it $\vec{u}_{d+1}$ then since we have the relationship $(A - \lambda I)\vec{u}_{m+1} = \vec{u}_m$ we can multiply $\vec{u}_{d+1}$ by $A - \lambda I$ to get the next generalized eigen vector $\vec{u}_d$. Then we can do the same thing to $\vec{u}_d$ to get $\vec{u}_{d-1}$ all the way down until we get $(A - \lambda I)\vec{u}_1 = 0$ at which point we know we have hit a regular eigen vector as we got zero so that $\vec{u}_1$ is a regular eigen vector $\vec{v}_1$

(1) Try to solve $(A - \lambda I)^{d+1} \vec{u}_{d+1} = 0$ if you get a solution then do

(2) $(A - \lambda I) \vec{u}_{d+1} = \vec{u}_d$

$(A - \lambda I) \vec{u}_d = \vec{u}_{d-1}$

$\vdots$

$(A - \lambda I) \vec{u}_2 = \vec{u}_1$

$(A - \lambda I) \vec{u}_1 = 0$

(3) once you get to zero the set $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_{d+1}\}$ is the chain for λ with $\vec{u}_1$ being a regular eigen vector of A .

If you cannot solve the equation in (1) move down from $d + 1$ to d , if you can't solve it with d step down until you can.

This last line in the algorithm is how we can tell between a situation like $\{\vec{v}_1, \vec{u}_1\}, \{\vec{v}_2, \vec{u}_2\}$ or $\{\vec{v}_1, \vec{u}_1, \vec{u}_2\}, \{\vec{v}_2\}$ If we started with $d + 1 = 3$ and we get a solution to the equation in step (1) we would be in the situation $\{\vec{v}_1, \vec{u}_1, \vec{u}_2\}, \{\vec{v}_2\}$. If not we would step down to d and solve $(A - \lambda I)^2 \vec{u}_2 = 0$ and we then know we would be in situation $\{\vec{v}_1, \vec{u}_1\}, \{\vec{v}_2, \vec{u}_2\}$ and we would get two different solutions to form the chains from .

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We know we will always be able to build a solution with the needed number of linearly independent vectors because of these two facts from linear algebra

- (1) Any chain of generalized eigen vectors will be a linearly independent set of vectors
- (2) If two chains of generalized eigen vectors are based on linearly independent eigen vectors then their union will be a linearly independent set of vectors.

So if we get a k chain $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_k\}$ of eigen vectors we will have a contribution to the full general solution of the linear system of k functions of the form

$$\begin{aligned}\vec{x}_1 &= \vec{u}_1 e^{\lambda t} \\ \vec{x}_2 &= (\vec{u}_1 t + \vec{u}_2) e^{\lambda t} \\ \vec{x}_3 &= \left(\frac{1}{2} \vec{u}_1 t^2 + \vec{u}_2 t + \vec{u}_3\right) e^{\lambda t} \\ &\vdots \\ \vec{x}_k &= \left(\frac{1}{(k-1)!} \vec{u}_1 t^{k-1} + \frac{1}{(k-2)!} \vec{u}_2 t^{k-2} + \frac{1}{(k-3)!} \vec{u}_3 t^{k-3} + \dots + \vec{u}_{k-1} t + \vec{u}_k\right) e^{\lambda t}\end{aligned}$$

So that $\vec{x}(t) = \text{other parts from other eigen vals} + c_1 \vec{x}_1 + c_2 \vec{x}_2 + \dots + c_k \vec{x}_k$

What would be the structure of the contributions of the eigen value with multiplicity 4 and defect 2 for the two possible situations of $\{\vec{v}_1, \vec{u}_1\}, \{\vec{v}_2, \vec{u}_2\}$ or $\{\vec{v}_1, \vec{u}_1, \vec{u}_2\}, \{\vec{v}_2\}$?

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Let's do an example with the help of octave

$$\vec{x}' = \begin{bmatrix} 0 & 1 & 2 \\ -5 & -3 & -7 \\ 1 & 0 & 0 \end{bmatrix} \vec{x}$$