

Section 5.5 Multiple Eigen Value Solutions

$$\frac{d\vec{x}}{dt} = A\vec{x} - e^{At}$$

In Section 5.3 we stated that when we have a repeated eigen value of a matrix there may or may not be more than one eigen vector associated with it.

Let's consider a system of two linear first order equations of the form

$$\vec{x}' = A\vec{x}$$

And let's suppose that we have a repeated eigen value of λ that only has one eigen vector associated with it $\vec{v}_1$

We said that in this case the solution to the system is

$$\vec{x} = c_1 \vec{v}_1 e^{\lambda t} + c_2 (\vec{v}_1 t + \vec{v}_2) e^{\lambda t}$$

$$A\vec{v}_1 = \lambda \vec{v}_1$$

Where $\vec{v}_2$ is called a generalized eigen vector. Let's see how we might find this eigen vector by plugging it in to the differential equation itself

LHS

$$\begin{aligned} \vec{x}' &= \frac{d}{dt} (c_1 \vec{v}_1 e^{\lambda t} + c_2 \vec{v}_1 t e^{\lambda t} + c_2 \vec{v}_2 e^{\lambda t}) \\ &= \lambda c_1 \vec{v}_1 e^{\lambda t} + c_2 \vec{v}_1 e^{\lambda t} + \lambda c_2 \vec{v}_1 t e^{\lambda t} + \lambda c_2 \vec{v}_2 e^{\lambda t} \end{aligned}$$

RHS $A\vec{x}$

$$\begin{aligned} &A(c_1 \vec{v}_1 e^{\lambda t} + c_2 \vec{v}_1 t e^{\lambda t} + c_2 \vec{v}_2 e^{\lambda t}) \\ &= c_1 A\vec{v}_1 e^{\lambda t} + c_2 A\vec{v}_1 t e^{\lambda t} + c_2 A\vec{v}_2 e^{\lambda t} \\ &= c_1 \lambda \vec{v}_1 e^{\lambda t} + c_2 \lambda \vec{v}_1 t e^{\lambda t} + c_2 A\vec{v}_2 e^{\lambda t} \end{aligned}$$

$$\cancel{\lambda c_1 \vec{v}_1 e^{\lambda t}} + c_2 \vec{v}_1 e^{\lambda t} + \cancel{\lambda c_2 \vec{v}_1 t e^{\lambda t}} + \lambda c_2 \vec{v}_2 e^{\lambda t} = \cancel{c_1 \lambda \vec{v}_1 e^{\lambda t}} + \cancel{c_2 \lambda \vec{v}_1 t e^{\lambda t}} + c_2 A\vec{v}_2 e^{\lambda t}$$

$$c_2 \vec{v}_1 e^{\lambda t} + \lambda c_2 \vec{v}_2 e^{\lambda t} = c_2 A\vec{v}_2 e^{\lambda t}$$

divide both sides by c_2 and $e^{\lambda t}$

$$\vec{v}_1 + \lambda \vec{v}_2 = A\vec{v}_2$$

$$\vec{v}_1 = A\vec{v}_2 - \lambda \vec{v}_2$$

$$(A - \lambda I)\vec{v}_2 = \vec{v}_1$$

To get $\vec{v}_2$ solve for $\vec{v}_2$

Since we know $\vec{v}_1$

So for the $\vec{x}(t)$ above to be a solution the generalized eigen vector $\vec{v}_2$ must satisfy $(A - \lambda I)\vec{v}_2 = \vec{v}_1$

Section 5.5 Multiple Eigen Value Solutions

Let's consider the system: $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 1 & -3 \\ 3 & 7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ and find the general solution and the generalized eigen vector.

Find λ 's $\det(A - \lambda I) = 0$ solve for λ

$$\begin{vmatrix} 1-\lambda & -3 \\ 3 & 7-\lambda \end{vmatrix} = (1-\lambda)(7-\lambda) - (-9) = 0$$

$$= \lambda^2 - 8\lambda + 7 + 9 = 0 \rightarrow \lambda^2 - 8\lambda + 16 = 0$$

$$(\lambda - 4)^2 = 0$$

② Find eigen vecs

$$A - 4I = \begin{bmatrix} -3 & -3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} -3 & -3 & | & 0 \\ 3 & 3 & | & 0 \end{bmatrix} \xrightarrow{R_1 + R_2} \begin{bmatrix} -3 & -3 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} -3 & -3 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \rightarrow -3x_1 - 3x_2 = 0 \rightarrow \underline{x_1 = -x_2}$$

$$\rightarrow 0x_1 + 0x_2 = 0$$

$$x_2 = s$$

$$x_2 = 1 \quad x_1 = -1 \rightarrow \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \vec{v}_1$$

③ Find generalized vectors

$$(A - \lambda I) \vec{v}_2 = \vec{v}_1 \rightarrow \begin{bmatrix} -3 & -3 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} -3 & -3 & | & -1 \\ 3 & 3 & | & 1 \end{bmatrix} \xrightarrow{R_1 + R_2} \begin{bmatrix} -3 & -3 & | & -1 \\ 0 & 0 & | & 0 \end{bmatrix}$$

$$\begin{aligned} -3x_1 - 3x_2 &= -1 \\ -3x_1 - 3s &= -1 \\ -3x_1 &= -1 + 3s \\ x_1 &= \frac{1}{3} - s \\ x_2 &= s \end{aligned}$$

$$\text{choose } s = 0 \rightarrow \begin{bmatrix} 1/3 \\ 0 \end{bmatrix} \rightarrow 3 \begin{bmatrix} 1/3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \vec{v}_2$$

$$\vec{x}(t) = c_1 \vec{v}_1 e^{\lambda t} + c_2 (\vec{v}_1 t + \vec{v}_2) e^{\lambda t}$$

$$\underline{\vec{x}(t)} = c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} e^{4t} + c_2 \left(\begin{bmatrix} -1 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) e^{4t} = \begin{bmatrix} (-c_1 - c_2 t + c_2) e^{4t} \\ (c_1 + c_2) e^{4t} \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

Section 5.5 Multiple Eigen Value Solutions

Let's just consider the general case of a repeated eigen value for a matrix A . We know that the way we find eigen values is to solve the characteristic polynomial coming from the equation

$$\det(A - \lambda I) = 0$$

For an $n \times n$ matrix A we know we could factor that equation into a product of linear terms involving the eigen values.

$$(\lambda - \lambda_1)(\lambda - \lambda_2) \cdots (\lambda - \lambda_n) = 0$$

If we have a repeated eigen value this might look more like

$$(\lambda - \lambda_1)^m (\lambda - \lambda_2) \cdots (\lambda - \lambda_{n-k}) = 0$$

Here we would say that the eigen value λ_1 would have algebraic multiplicity m . If when solving for the eigen vectors for λ_1 we find m linearly independent ones we say that λ_1 is **complete**. If there are only $p < m$ linearly independent eigen vectors we call it a **defective** eigen value. Further we call the number of missing eigen vectors $d = m - p$ the **defect**. This will also be the number of generalized eigen vectors that we can find.

We can find all of the generalized eigen vectors belonging to a defective eigen value successively. Suppose that $\vec{v}_1$ is one of the regular eigen vectors of λ, A which means that it satisfies.

$$(A - \lambda I)\vec{v}_1 = 0$$

From what we saw before a generalized eigen vector can be found solving

$$(A - \lambda I)\vec{v}_2 = \vec{v}_1$$

for $\vec{v}_2$. The next generalized eigen vector may be found with

$$(A - \lambda I)\vec{v}_3 = \vec{v}_2$$

All the way up to how ever many generalized eigen vectors might be generated from $\vec{v}_1$

$$\vdots$$

$$(A - \lambda I)\vec{v}_k = \vec{v}_{k-1}$$

Note the following pattern using the above sequence:

$$(A - \lambda I)\vec{v}_2 = \vec{v}_1$$

$$(A - \lambda I)(A - \lambda I)\vec{v}_2 = (A - \lambda I)\vec{v}_1 = 0 \rightarrow (A - \lambda I)^2\vec{v}_2 = 0$$

then

$$(A - \lambda I)^2(A - \lambda I)\vec{v}_3 = (A - \lambda I)^2\vec{v}_2 = 0 \rightarrow (A - \lambda I)^3\vec{v}_3 = 0$$

And so on until

$$(A - \lambda I)^k\vec{v}_k = 0$$

Here we would say that k is the rank of the generalized eigen vector.

$$A\vec{v}_1 = \lambda\vec{v}_1$$

$$(A - \lambda I)(A - \lambda I)\vec{v}_2 = (A - \lambda I)\vec{v}_1 = 0$$

$$(A - \lambda I)\vec{v}_3 = \vec{v}_2$$

$$(A - \lambda I)^2(A - \lambda I)\vec{v}_3 = (A - \lambda I)^2\vec{v}_2 = 0$$

$$= (A - \lambda I)^2\vec{v}_2$$

Section 5.5 Multiple Eigen Value Solutions

We would call the set of vectors we found $\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_k\}$ a length k chain of generalized eigen vectors based on the eigen vector $\vec{v}_1$. *Honest*

It may not be that we get all of our generalized eigen vectors from $\vec{v}_1$ however. If we have other regular eigen vectors coming from lambda they may also be generators of some of the generalized eigen vectors of λ and A .

Let's choose a different notation for generalized eigen vectors now, let's denote them with a u and the regular eigen vectors with a v .

To illustrate the possibilities with a defective eigen value and its generalized eigen vectors let's suppose the matrix A has an eigen value λ with multiplicity $m = 4$ and defect $d = \dots$.

$d = 0$, Then we have $\{\vec{v}_1\}, \{\vec{v}_2\}, \{\vec{v}_3\}, \{\vec{v}_4\}$ four length 1 chains made up of regular eigen vectors

$d = 1$, Then we have $\{\vec{v}_1, \vec{u}_1\}, \{\vec{v}_2\}, \{\vec{v}_3\}$ two length 1 chains of regular eigen vectors and a length 2 chain of a regular eigen vector and a generalized eigen vector coming from it.

$d = 2$ Then we can have $\{\vec{v}_1, \vec{u}_1\}, \{\vec{v}_2, \vec{u}_2\}$ two length two chains with a regular eigen vector and a generalized eigen vector generated from each.

$d = 2$ We could also have $\{\vec{v}_1, \vec{u}_1, \vec{u}_2\}, \{\vec{v}_2\}$ One length 3 chain and one length 1 chain.

$d = 3$ $\{\vec{v}_1, \vec{u}_1, \vec{u}_2, \vec{u}_3\}$ a length 4 chain with generalized eigen vectors all coming from the only regular one found.

Suppose we indeed have an eigen value λ with multiplicity 4 and defect 2, how can we know whether the vectors we need come in the form $\{\vec{v}_1, \vec{u}_1\}, \{\vec{v}_2, \vec{u}_2\}$ or $\{\vec{v}_1, \vec{u}_1, \vec{u}_2\}, \{\vec{v}_2\}$ Well we will be able to know but first let's give an algorithm for constructing the chains... Also head here for a nice set of notes going through two examples like this... https://www.math.purdue.edu/~neptamin/303Au21/Handouts/High_defect.pdf

We notice that in every case above the length of the chain was at most $d + 1$ so we could begin looking for generalized eigen vectors by solving

$$(A - \lambda I)^{d+1} \vec{u}_{d+1} = 0$$

and

$$(A - \lambda I)^d \vec{u}_{d+1} \neq 0$$

If we get a solution call it $\vec{u}_{d+1}$ then since we have the relationship $(A - \lambda I) \vec{u}_{m+1} = \vec{u}_m$ we can multiply $\vec{u}_{d+1}$ by $A - \lambda I$ to get the next generalized eigen vector $\vec{u}_d$. Then we can do the same thing to $\vec{u}_d$ to get $\vec{u}_{d-1}$ all the way down until we get $(A - \lambda I) \vec{u}_1 = 0$ at which point we know we have hit a regular eigen vector as we got zero so that $\vec{u}_1$ is a regular eigen vector $\vec{v}_1$

(1) Try to solve $(A - \lambda I)^{d+1} \vec{u}_{d+1} = 0, (A - \lambda I)^d \vec{u}_{d+1} \neq 0$ if you get a solution then do

(2) $(A - \lambda I) \vec{u}_{d+1} = \vec{u}_d$

$(A - \lambda I) \vec{u}_d = \vec{u}_{d-1}$

$(A - \lambda I) \vec{u}_2 = \vec{u}_1$

$(A - \lambda I) \vec{u}_1 = 0$

(3) once you get to zero the set $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_{d+1}\}$ is the chain for λ with $\vec{u}_1$ being a regular eigen vector of A .

If you cannot solve the equations in (1) move down from $d + 1$ to d , if you can't solve it with d step down until you can.

This last line in the algorithm is how we can tell between a situation like $\{\vec{v}_1, \vec{u}_1\}, \{\vec{v}_2, \vec{u}_2\}$ or $\{\vec{v}_1, \vec{u}_1, \vec{u}_2\}, \{\vec{v}_2\}$ If we started with $d + 1 = 3$ and we get a solution to the equation in step (1) we would be in the situation $\{\vec{v}_1, \vec{u}_1, \vec{u}_2\}, \{\vec{v}_2\}$. If not we would step down to d and solve $(A - \lambda I)^2 \vec{u}_2 = 0$ and we then know we would be in situation $\{\vec{v}_1, \vec{u}_1\}, \{\vec{v}_2, \vec{u}_2\}$ and we would get two different solutions to form the chains from.

Section 5.5 Multiple Eigen Value Solutions

We know we will always be able to build a solution with the needed number of linearly independent vectors because of these two facts from linear algebra

- (1) Any chain of generalized eigen vectors will be a linearly independent set of vectors
- (2) If two chains of generalized eigen vectors are based on linearly independent eigen vectors then their union will be a linearly independent set of vectors.

So if we get a k chain $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_k\}$ of eigen vectors we will have a contribution to the full general solution of the linear system of k functions of the form

$$\begin{aligned} \vec{x}_1 &= \vec{u}_1 e^{\lambda t} \\ \vec{x}_2 &= (\vec{u}_1 t + \vec{u}_2) e^{\lambda t} \\ \vec{x}_3 &= \left(\frac{1}{2} \vec{u}_1 t^2 + \vec{u}_2 t + \vec{u}_3 \right) e^{\lambda t} \\ &\vdots \\ \vec{x}_k &= \left(\frac{1}{(k-1)!} \vec{u}_1 t^{k-1} + \frac{1}{(k-2)!} \vec{u}_2 t^{k-2} + \frac{1}{(k-3)!} \vec{u}_3 t^{k-3} + \dots + \vec{u}_{k-1} t + \vec{u}_k \right) e^{\lambda t} \end{aligned}$$

So that $\vec{x}(t) = \text{other parts from other eigen vals} + c_1 \vec{x}_1 + c_2 \vec{x}_2 + \dots + c_k \vec{x}_k$

What would be the structure of the contributions of the eigen value with multiplicity 4 and defect 2 for the two possible situations of $\{\vec{v}_1, \vec{u}_1\}, \{\vec{v}_2, \vec{u}_2\}$ or $\{\vec{v}_1, \vec{u}_1, \vec{u}_2\}, \{\vec{v}_2\}$?

$$\begin{aligned} & c_1 \vec{v}_1 e^{\lambda t} + c_2 (\vec{v}_1 t + \vec{u}_1) e^{\lambda t} \quad c_3 \vec{v}_2 e^{\lambda t} + c_4 (\vec{v}_2 t + \vec{u}_2) e^{\lambda t} \\ & \text{from 2 chain} \quad \text{from 2 chain} \\ & \{\vec{v}_1, \vec{u}_1\} \quad \{\vec{v}_2, \vec{u}_2\} \end{aligned}$$

$$\begin{aligned} & c_1 \vec{v}_1 e^{\lambda t} + c_2 (\vec{v}_1 t + \vec{u}_1) e^{\lambda t} + c_3 \left(\frac{1}{2} \vec{v}_1 t^2 + \vec{u}_1 t + \vec{u}_2 \right) e^{\lambda t} + c_4 \vec{v}_2 e^{\lambda t} \end{aligned}$$

Section 5.5 Multiple Eigen Value Solutions

Let's do an example with the help of octave

$$\vec{x}' = \begin{bmatrix} 0 & 1 & 2 \\ -5 & -3 & -7 \\ 1 & 0 & 0 \end{bmatrix} \vec{x}$$

$$\lambda = -1 \quad m = 3 \quad d = 2$$

$$(A - \lambda I)^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow (A - \lambda I)^3 \vec{v} = 0$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\text{choose } \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \vec{v}_3$$

$$\text{check } (A - \lambda I)^2 \vec{v}_3 \neq 0$$

$$\rightarrow (A - \lambda I) \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -5 \\ 1 \end{bmatrix} = \vec{v}_2$$

$$(A - \lambda I) \vec{v}_2 = \vec{v}_1 = \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix} \quad \text{Honest eigen vec}$$

$$\{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \} =$$

$$x(t) = c_1 \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix} e^{-t} + c_2 \left(\begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix} t + \begin{bmatrix} 1 \\ -5 \\ 1 \end{bmatrix} \right) e^{-t} + c_3 \left(\frac{1}{2!} \begin{bmatrix} -2 \\ -2 \\ 2 \end{bmatrix} t^2 + \begin{bmatrix} 1 \\ -5 \\ 1 \end{bmatrix} t + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right) e^{-t}$$

Section 5.5 Multiple Eigen Value Solutions

Octave or matlab code to get there

```
>> A=[0 1 2; -5 -3 -7; 1 0 0]
```

A =

```
0    1    2
-5   -3   -7
1    0    0
```

```
>> [vecs,lams]=eig(A)
```

vecs = $\vec{v}_1$ $\vec{v}_2$ $\vec{v}_3$

$$\begin{bmatrix} 0.5773 + 0.0000i \\ 0.5774 + 0.0000i \\ -0.5773 + 0.0000i \end{bmatrix} \begin{bmatrix} 0.5774 + 0.0000i \\ 0.5773 + 0.0000i \\ -0.5774 - 0.0000i \end{bmatrix} \begin{bmatrix} 0.5774 + 0.0000i \\ 0.5773 - 0.0000i \\ -0.5774 + 0.0000i \end{bmatrix} \rightarrow \text{only one eigen vector } \vec{v}_1 = \vec{v}_2 = \vec{v}_3$$

so detect $d=3-1=2$

lams = $\begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \rightarrow \lambda_1, \lambda_2 = \lambda_3 = -1$

```
-1.0000 + 0.0000i    0.0000 + 0.0000i    0.0000 + 0.0000i
0.0000 + 0.0000i   -1.0000 + 0.0000i    0.0000 + 0.0000i
0.0000 + 0.0000i    0.0000 + 0.0000i   -1.0000 - 0.0000i
```

$\rightarrow \lambda = -1$ multiplicity 3

```
>> (A-(-1)*eye(3))^3
```

ans =

```
0    0    0
0    0    0
0    0    0
```

$\rightarrow$ solve $(A-\lambda I)^{d+1} \vec{v}_{d+1} = 0 \Rightarrow (A-\lambda I)^3 \vec{v}_3 = 0$

Since $(A-(-1)I)^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$ for any vector $\vec{v}_1$

so we can choose $\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

```
>> u3=[1;0;0]
```

u3 =

```
1
0
0
```

```
>> (A-(-1)*eye(3))*u3
```

ans =

```
1
-5
1
```

```
>> u2=[1; -5; 1]
```

u2 =

```
1
-5
1
```

$\rightarrow$ get $\vec{v}_2$ with $(A-\lambda I)\vec{v}_3 = \vec{v}_2$ which works for $\vec{v}_2 \neq 0$ If $(A-\lambda I)\vec{v}_3 = 0$ restart alone with $(A-\lambda I)^d \vec{v}_d = 0$

```
>> (A-(-1)*eye(3))*u2
```

ans =

```
-2
-2
2
```

note $(A-\lambda I)\vec{v}_2 = \vec{v}_1$

$\leftarrow$ gives us a multiple of our lowest eigen value!