

Section 5.7 Non-Homogeneous Linear Systems

Back in section 3.5 we learned how to solve non-homogeneous linear equations of the form

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_2 y'' + a_1 y' + a_0 y = f(x)$$

We essentially had two methods to do this, The Method Of Undetermined Coefficients and Variation of Parameters

As it turns out we can generalize these methods to solve linear systems of non-homogeneous equations.

$$\begin{cases} x_1' = a_{11}x_1 + a_{21}x_2 + \dots + a_{n1}x_n + f_1(t) \\ x_2' = a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n + f_2(t) \\ \vdots \\ x_n' = a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n + f_n(t) \end{cases}$$

Which can also be written in the form

$$\frac{d\vec{x}}{dt} = A\vec{x} + \vec{f}(t)$$

We will focus on just the method of undetermined coefficients however since we would need more on the matrix exponential to do variation of parameters.

Just as before we will build the full solution out of the complementary solution $\vec{x}_c$ which solves the homogeneous system

$$\begin{cases} x_1' = a_{11}x_1 + a_{21}x_2 + \dots + a_{n1}x_n \\ x_2' = a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n \\ \vdots \\ x_n' = a_{n1}x_1 + a_{n2}x_2 + \dots + a_{nn}x_n \end{cases}$$

And the particular solution $\vec{x}_p$ which solves the full system with the $f(t)$'s above. Note that these are now just vector solutions. Incidentally the method of undetermined coefficients plays out in a very similar way as before except instead of undetermined scalar coefficient we need to find undetermined constant vectors!

Example: Consider the system of equations $\vec{x}' = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} \vec{x} + \begin{bmatrix} 3 \\ 2t \end{bmatrix}$ What would be our guess of $\vec{x}_p$?

$$x_p = \vec{a}_1 t + \vec{a}_2 = \begin{bmatrix} a_{1,1} \\ a_{1,2} \end{bmatrix} t + \begin{bmatrix} a_{2,1} \\ a_{2,2} \end{bmatrix}$$

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Let's find the full solution to the system... $\vec{x}' = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} \vec{x} + \begin{bmatrix} 3 \\ 2t \end{bmatrix}$ $\begin{bmatrix} a_{11} \\ a_{12} \end{bmatrix} t + \begin{bmatrix} a_{21} \\ a_{22} \end{bmatrix} = \vec{x}_p$

Find $\vec{x}_c$ soln to $\vec{x}' = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} \vec{x}$

Find eigen stuff

eig vals
 $|(\mathbf{A} - \lambda \mathbf{I})| = 0 \quad \begin{vmatrix} 3-\lambda & 2 \\ 7 & 5-\lambda \end{vmatrix} = (3-\lambda)(5-\lambda) - 14 = \lambda^2 - 8\lambda + 15 - 14 = \lambda^2 - 8\lambda + 1 = 0$
 $\lambda^2 - 8\lambda + 1 = 0 \rightarrow \frac{8 \pm \sqrt{64 - 4(1)(1)}}{2} = \frac{8 \pm \sqrt{60}}{2} = \frac{8 \pm 2\sqrt{15}}{2}$

$4 \pm \sqrt{15} \quad \lambda_1 = 4 + \sqrt{15} \quad \lambda_2 = 4 - \sqrt{15}$

Then solve $\begin{bmatrix} 3 - (4 + \sqrt{15}) & 2 \\ 7 & 5 - (4 + \sqrt{15}) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \vec{v}_1, \lambda_1$

$\begin{bmatrix} 3 - (4 - \sqrt{15}) & 2 \\ 7 & 5 - (4 - \sqrt{15}) \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \rightarrow \vec{v}_2, \lambda_2$

$\vec{x}_c = c_1 \vec{v}_1 e^{(4 + \sqrt{15})t} + c_2 \vec{v}_2 e^{(4 - \sqrt{15})t}$

$\vec{x}' = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} \vec{x} + \begin{bmatrix} 3 \\ 2t \end{bmatrix} \quad \vec{x}_p = \begin{bmatrix} a_{11} \\ a_{12} \end{bmatrix} t + \begin{bmatrix} a_{21} \\ a_{22} \end{bmatrix} = \begin{bmatrix} a_{11}t + a_{21} \\ a_{12}t + a_{22} \end{bmatrix}$

$\begin{bmatrix} a_{11} \\ a_{12} \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 7 & 5 \end{bmatrix} \begin{bmatrix} a_{11}t + a_{21} \\ a_{12}t + a_{22} \end{bmatrix} + \begin{bmatrix} 3 \\ 2t \end{bmatrix}$

$\begin{bmatrix} a_{11} \\ a_{12} \end{bmatrix} = \begin{bmatrix} 3a_{11}t + 3a_{21} + 2a_{12}t + 2a_{22} \\ 7a_{11}t + 7a_{21} + 5a_{12}t + 5a_{22} \end{bmatrix} + \begin{bmatrix} 3 \\ 2t \end{bmatrix}$

$\begin{bmatrix} a_{11} \\ a_{12} \end{bmatrix} = \begin{bmatrix} (3a_{11} + 2a_{12})t + (3a_{21} + 2a_{22}) \\ (7a_{11} + 5a_{12})t + (7a_{21} + 5a_{22}) \end{bmatrix} + \begin{bmatrix} 3 \\ 2t \end{bmatrix}$

$\begin{bmatrix} -3 \\ -2t \end{bmatrix} = \begin{bmatrix} (3a_{11} + 2a_{12})t + (3a_{21} + 2a_{22} - a_{11}) \\ (7a_{11} + 5a_{12})t + (7a_{21} + 5a_{22} - a_{12}) \end{bmatrix}$

$3a_{11} + 2a_{12} = 0$

$3a_{21} + 2a_{22} - a_{11} = -3$

$7a_{11} + 5a_{12} = -2$

$7a_{21} + 5a_{22} - a_{12} = 0$

$\mathbf{B} = \begin{bmatrix} 3 & 2 & 0 & 0 & 0 \\ -1 & 0 & 3 & 2 & -3 \\ 7 & 5 & 0 & 0 & -2 \\ 0 & -1 & 7 & 5 & 0 \end{bmatrix}$

give to octave
 $\text{rref}(\mathbf{B})$

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In octave its easy to solve!

```
octave:3> B=[3 2 0 0 0; -1 0 3 2 -3; 7 5 0 0 -2; 0 -1 7 5 0]
```

B =

```
3  2  0  0  0
-1 0  3  2 -3
7  5  0  0 -2
0 -1  7  5  0
```

```
octave:4> rref(B)
```

ans =

a_{11}	a_{12}	a_{21}	a_{22}	
1.0000	0	0	0	4.0000 $\rightarrow a_{11} = 4$
0	1.0000	0	0	-6.0000 $\rightarrow a_{12} = -6$
0	0	1.0000	0	17.0000 $\rightarrow a_{21} = 17$
0	0	0	1.0000	-25.0000 $\rightarrow a_{22} = -25$

so

$$\vec{x}_p = \begin{bmatrix} 4 \\ -6 \end{bmatrix} t + \begin{bmatrix} 17 \\ -25 \end{bmatrix}$$

$$\text{and } \vec{x}(t) = \vec{x}_h(t) + \vec{x}_p(t)$$

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While this method is very similar to the method of undetermined coefficients just replacing constants with vector constants there is one difference. When we have duplication we must not only multiply by the smallest power of s but by all lower (non zero) integral powers of t as well and we include all of them in the guess.

Given the system $x' = \begin{bmatrix} 4 & 2 \\ 3 & -1 \end{bmatrix} x - \begin{bmatrix} 15 \\ 4 \end{bmatrix} e^{-2t}$ has complementary solution $x_c = c_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} e^{-2t} + c_2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{5t}$

$$\begin{vmatrix} 4-\lambda & 2 \\ 3 & -1-\lambda \end{vmatrix} = (4-\lambda)(-1-\lambda) - 6 = \lambda^2 - 3\lambda - 4 - 6$$

$$\lambda^2 - 3\lambda - 10 = (\lambda + 2)(\lambda - 5) \rightarrow x_c = c_1 \vec{v}_1 e^{-2t} + c_2 \vec{v}_2 e^{5t}$$

$$\rightarrow x_p = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} e^{-2t} \quad \text{repeat}$$

$$x_p = \vec{a} e^{-2t} + \vec{b} e^{-2t}$$

If a term $g(t)$ that we are guessing with the particular solution is repeated in the complementary solution of our linear system we build the contribution to our guess for $\vec{x}_p$ from $g(t)$ to be:

$$\underline{t^s g(t)} + \underline{t^{s-1} g(t)} + \cdots + \underline{t g(t)} + \underline{g(t)}$$

Where s is the smallest integer such that $t^s g(t)$ is not a repeat of terms in the complementary solution

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Another example: Find the form of the particular solution to the system of equations but do not find the coefficients.

$$x' = x - 26y + 4\cos(5t)$$

$$y' = x - y$$

Find eigen values

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -26 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 4\cos(5t) \\ 0 \end{bmatrix}$$

$$\left| \begin{bmatrix} 1-\lambda & -26 \\ 1 & -1-\lambda \end{bmatrix} \right| = (1-\lambda)(-1-\lambda) + 26 = \lambda^2 + \lambda - 1 + 26 = \lambda^2 + 25 = 0$$

$$\Rightarrow \lambda = \pm\sqrt{-25} = \pm 5i$$

Find eigen vectors recall for complex $\vec{v}_1 = \overline{\vec{v}_2}$

$$\begin{bmatrix} 1-5i & -26 \\ 1 & -1-5i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow (1-5i)v_1 - 26v_2 = 0$$

$$v_1 - (1+5i)v_2 = 0 \Rightarrow v_1 = (1+5i)v_2$$

$$\Rightarrow (1-5i)(1+5i)v_2 - 26v_2 = 0$$

$$26v_2 - 26v_2 = 0$$

$$v_2 = v_2 \quad \text{any } v_2 \text{ works}$$

$$\text{choose } v_2 = 1$$

$$\text{then } v_1 = 1+5i$$

$$\text{Then } \vec{v}_1 = \begin{bmatrix} 1 \\ 1+5i \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} 1 \\ 1-5i \end{bmatrix}$$

for complex eigen vals

$$\vec{x}_1 = \text{Re}[\vec{v}_1 e^{\lambda_1 t}], \quad \vec{x}_2 = \text{Im}[\vec{v}_1 e^{\lambda_1 t}]$$

$$v_1 e^{\lambda_1 t} = \begin{bmatrix} 1 \\ 1+5i \end{bmatrix} e^{5it} = \begin{bmatrix} 1 \\ 1+5i \end{bmatrix} (\cos(5t) + i\sin(5t)) = \begin{bmatrix} \cos(5t) + i\sin(5t) \\ (1+5i)(\cos(5t) + i\sin(5t)) \end{bmatrix}$$

$$= \begin{bmatrix} \cos(5t) + i\sin(5t) \\ \cos(5t) + i\sin(5t) + 5i\cos(5t) - 5\sin(5t) \end{bmatrix} = \begin{bmatrix} \cos(5t) \\ \cos(5t) - 5\sin(5t) \end{bmatrix} + i \begin{bmatrix} \sin(5t) \\ \sin(5t) + 5\cos(5t) \end{bmatrix}$$

$$\text{so } \vec{x}_c = c_1 \begin{bmatrix} \cos(5t) \\ \cos(5t) - 5\sin(5t) \end{bmatrix} + c_2 \begin{bmatrix} \sin(5t) \\ \sin(5t) + 5\cos(5t) \end{bmatrix} \leftarrow \text{so we see } \vec{x}_p = \vec{a}_1 \cos(5t) + \vec{a}_2 \sin(5t) \text{ would have repeats...}$$

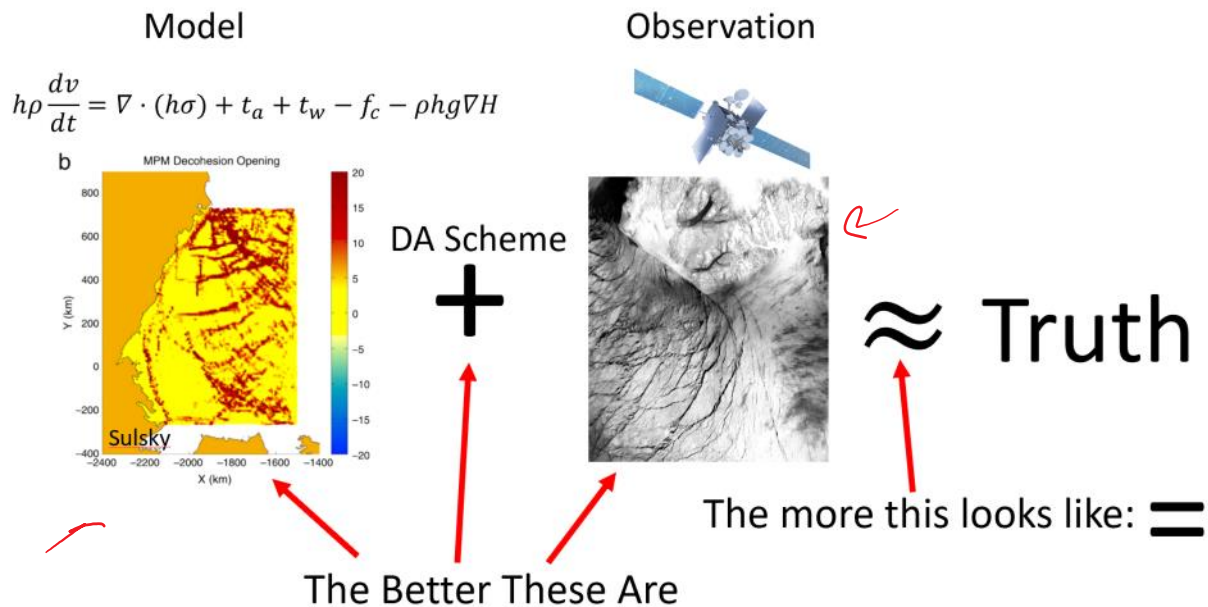
$$\text{so } \vec{x}_{1p} = \vec{a}_1 t \cos(5t) + \vec{a}_2 t \sin(5t) + \vec{a}_3 \cos(5t) + \vec{a}_4 \sin(5t)$$

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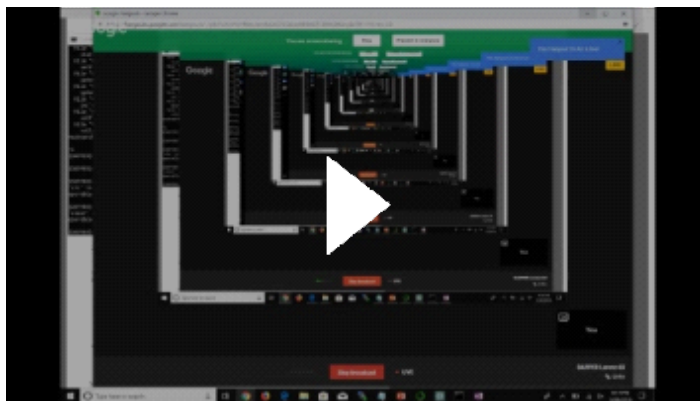
That's it! That's all we got!

I do just want to mention the field of data assimilation to y'all though. In modern times skills with large data sets for practical application means employability. Data Assimilation is one of these fields along with things like machine learning. Data Assimilation is the science of combining model (usually a differential equation) and real world data to produce a good estimate of the truth. In fact one of the reasons that the weather forecast on your phone is actually accurate these days is because Data Assimilation is done on a massive scale in conjunction with the complex global atmospheric models.

Data Assimilation



DAPPER Lorenz 63



If you are interested in learning more about DA and building some useful python coding skills I would absolutely recommend working through the DA tutorials coming out of the NANSSEN center in Bergen, Norway. You can find them here. <https://github.com/nansencenter/DA-tutorials>

On that github page you will find you can open them in google colab which means you won't have to even install python to get going!

