

Section 1.1 Linear Equations

Linear Equations show up in many areas of mathematics, natural and computer science. A linear equation is an equation of the form:

$$\underline{a_1x_1 + a_2x_2 + a_3x_3 + \dots + a_nx_n = b}$$

Where $\underline{a_1, a_2, a_3 \dots a_n}$ are constants and $\underline{x_1, x_2, x_3, \dots, x_n}$ are distinct variables.

Rapid fire: Linear or not?

→ $2s + 3t = 0$
 $\downarrow \quad \downarrow$
 $x_1 \quad x_2$
 Yes

$2x + 3y + 4z + 5w = x$
 $2x - x + 3y + 4z + 5w = 0$
 $x + 3y + 4z + 5w = 0$
 Yes

$3\sqrt{x} + 2y + z = 1$
 $\uparrow$
 $x^{1/2}$
 NO

$\frac{3x^2}{x} + 4y + 3z = 0$
 $x \neq 0$
 $3x + 4y + 3z = 0$

$y = 2x + 1$
 ① $y - 2x = 1$

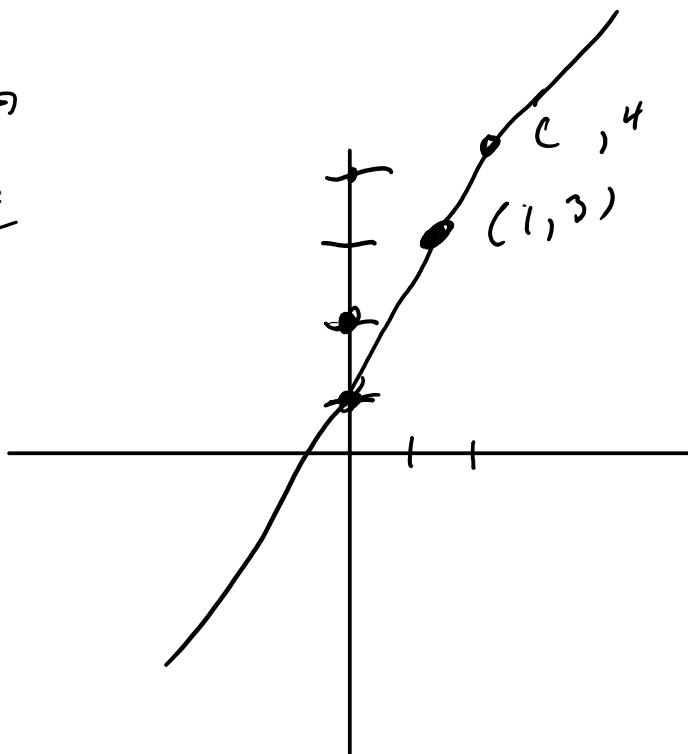
$4x^2 + 2y + 1 = 50$
 NO

Basically, it is an equation with only single powers of each variable multiplied by a given constant and the some of which is also constant.

Ex. $y - 2x = 1$ $(x, y) \in \mathbb{R}^2$

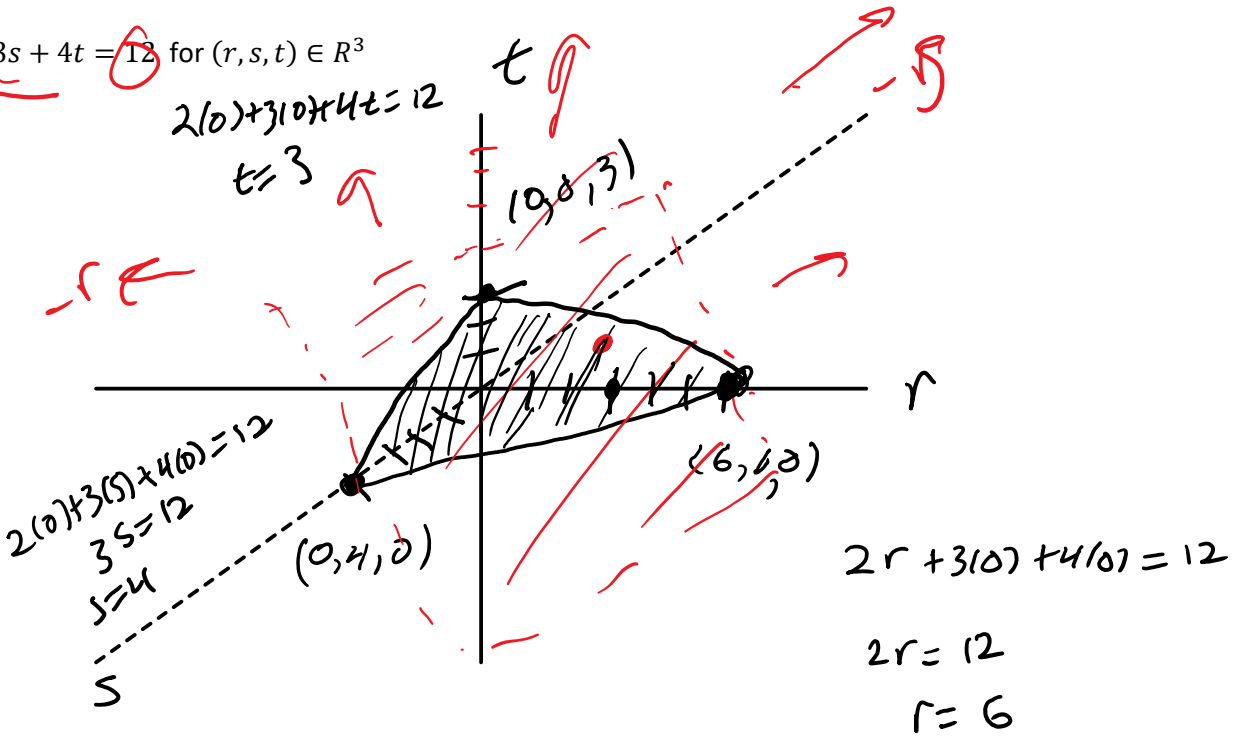
$y = 2x + 1$

$3 - 2(1) = 1$



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Ex $2r + 3s + 4t = 12$ for $(r, s, t) \in \mathbb{R}^3$



What do linear equations model? Too many things to name.

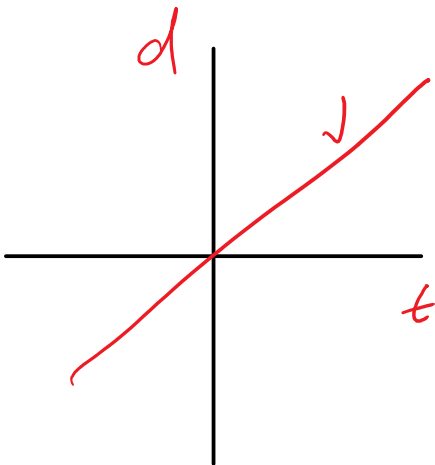
distance = velocity $\times$ time

Models that can be "linearized"

Archies law: $\sigma^* = \sigma_w \phi^m$

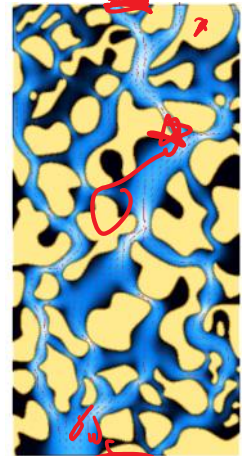
$$d = vt$$

$$d - vt = 0$$



$m \neq 1$

porosity



$$\ln(\sigma^*) = \ln(\sigma_w \phi^m)$$

$$\ln(\sigma^*) = \ln(\sigma_w) + m \ln(\phi)$$

$$\ln(\sigma^*) = \ln(\sigma_w) + m \ln(\phi)$$

$$\ln(\sigma^*) = m \ln(\phi) + \ln(\sigma_w)$$

$$y = mx + b$$



Section 1.1 Linear Equations

Systems of Linear equations:

A system of linear equations is a set of m linear equations with n variables

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\ \vdots & \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

A solution to a linear system is a n -tuple $(s_1, s_2, \dots, s_n)$ that satisfies every equation in the system. A solution set is the collection of solution points.

Example: An airline sold 100 coach seats for an upcoming flight from Raleigh to Cabo. Some were sold to students at a discount rate of \$50 round trip and to the general public for \$100 round trip. If the revenue from the sales was \$6000, how many students booked tickets?

S = students G = general public

$$S + G = 100 \rightarrow G = 100 - S$$

$$50S + 100G = 6000$$

$$50S + 100(100 - S) = 6000$$

$$50S + 10000 - 100S = 6000$$

$$-50S = -4000$$

$$S = \frac{-4000}{-50} = 80$$

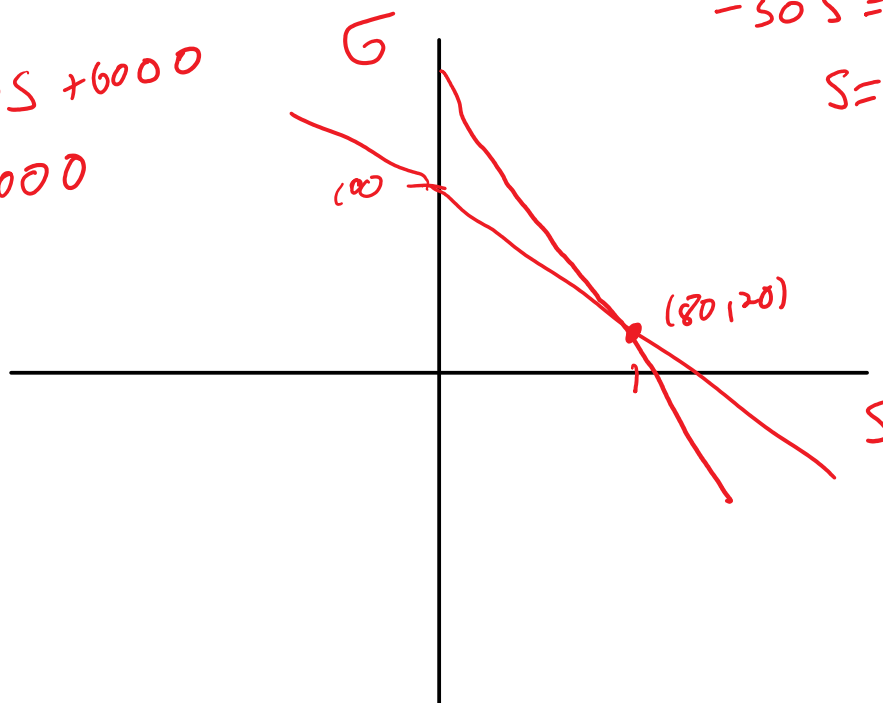
$$G = 20$$

$$S + G = 100$$

$$G = -S + 100$$

$$100G = -50S + 6000$$

$$G = -\frac{1}{2}S + 60$$



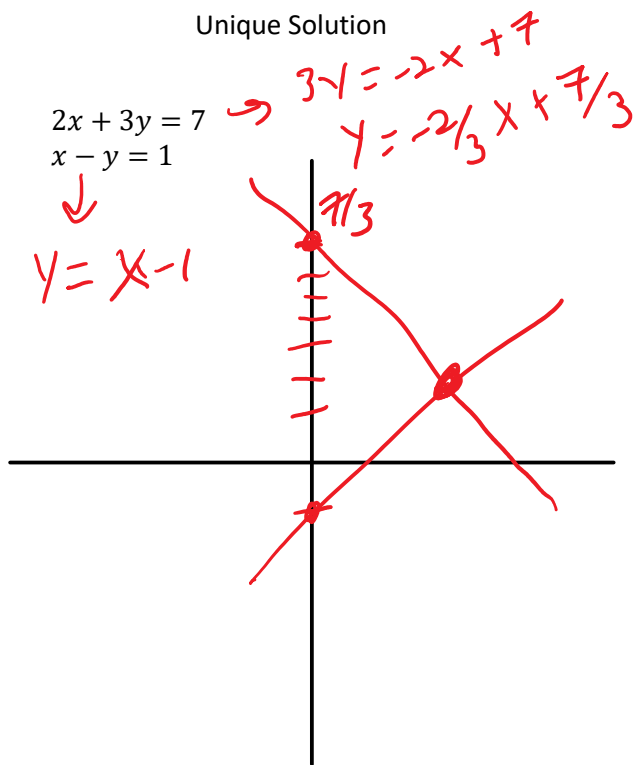
(S, G)
 $(80, 20)$

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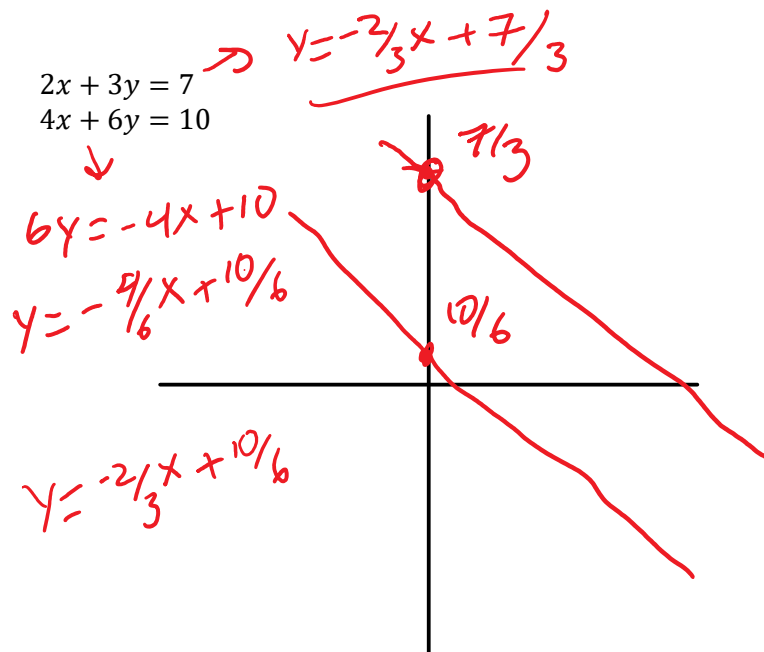
Number of Solutions

For lines

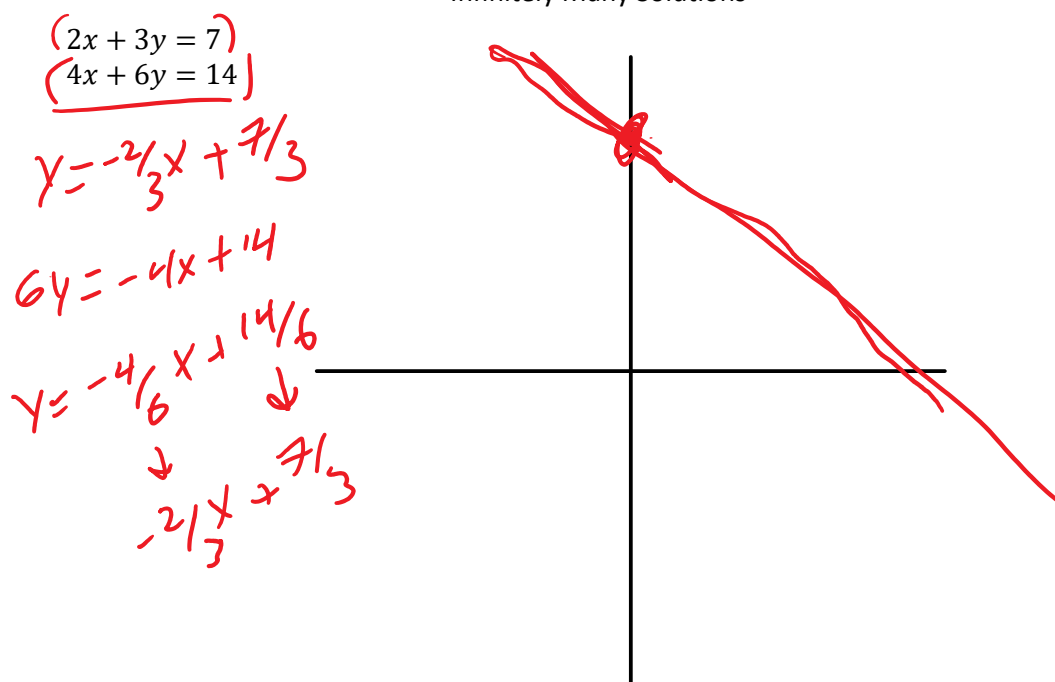
Unique Solution



No Solution

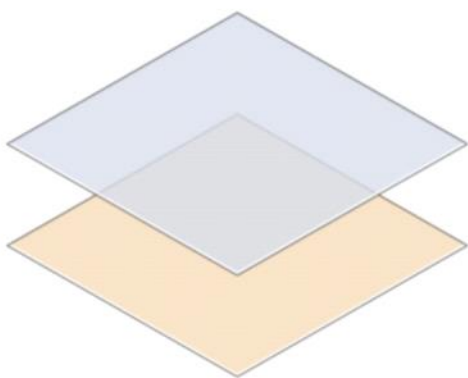


Infinitely Many Solutions

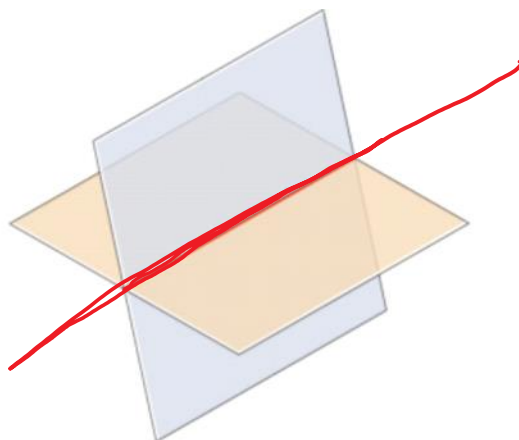


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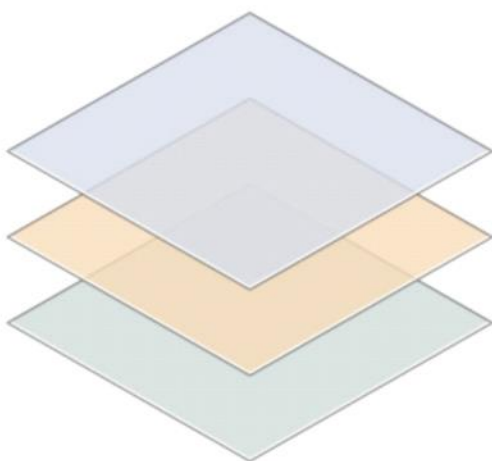
For Planes $ax + by + cz = d$



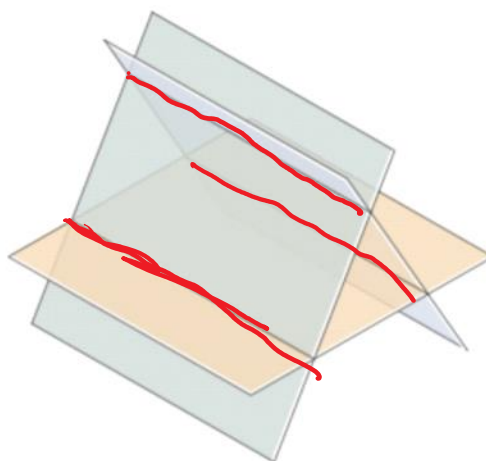
(a) Parallel planes, no points in common.



(b) Planes intersect, infinitely many points in common.

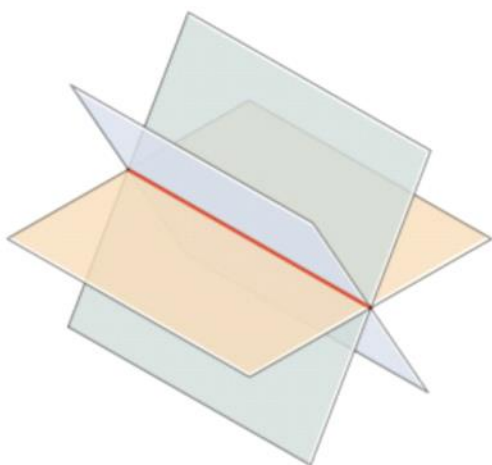


(a) Parallel planes, no points in common to all three.

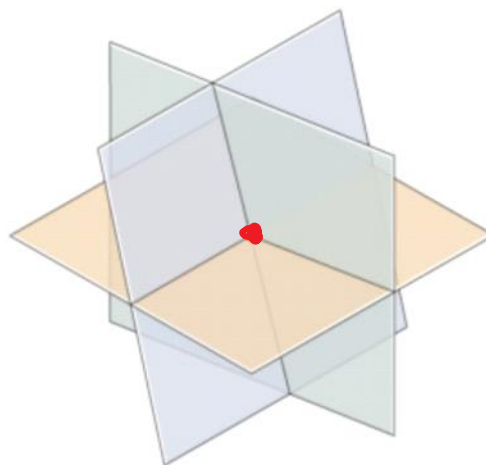


(b) Planes intersect in pairs, no points in common to all three.

NO
Solution



(c) Planes intersect in a line, infinitely many points in common.



(d) Planes intersect at a point, unique point in common.

Section 1.1 Linear Equations

Solving a Linear System (Elimination)

Handwritten solution for a system of linear equations using elimination:

$$\begin{aligned} 2x + 3y &= 1 \\ 4x + 2y &= 6 \end{aligned}$$

Step 1: Multiply the first equation by -2 to eliminate x.

$$-2(2x + 3y) = -2(1) \Rightarrow -4x - 6y = -2$$

Step 2: Add the two equations.

$$4x + 2y + (-4x - 6y) = 6 + (-2) \Rightarrow -4y = 4$$

Step 3: Solve for y.

$$y = -1$$

Step 4: Substitute y = -1 into the first equation to solve for x.

$$2x + 3(-1) = 1 \Rightarrow 2x - 3 = 1 \Rightarrow 2x = 4 \Rightarrow x = 2$$

Note that this was equivalent to the form

$$\rightarrow 2x + 3y = 1$$

$$\rightarrow -4y = 4$$

We call this triangular form and it is easier to solve if we can get there.

A triangular linear system has n variables and n equations and is of the form:

General form of a triangular linear system:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \cdots + a_{1n}x_n &= b_1 \\ a_{22}x_2 + a_{23}x_3 + \cdots + a_{2n}x_n &= b_2 \\ a_{33}x_3 + \cdots + a_{3n}x_n &= b_3 \\ \vdots &\vdots \\ a_{nn}x_n &= b_n \end{aligned}$$

Triangular systems have the following properties:

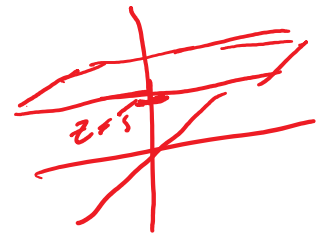
1. Every variable is the leading variable of exactly one equation
2. There are as many equations as variables
3. There is exactly one solution

Section 1.1 Linear Equations

Solving Triangular Systems

x_1 x_2

$$x_3 = 5$$



If we can put a linear system into a triangular form we can then use a process called back substitution to solve them.

Example: Solve the following triangular system

$$x_1 - 2x_2 - 5x_3 + 3x_4 = 2$$

$$x_2 + 3x_3 - 4x_4 = 7$$

$$x_3 + 2x_4 = -4$$

$$x_4 = 5$$

$$x_3 + 2(5) = -4$$

$$x_3 = -4 - 10$$

$$x_3 = -14$$

$$x_2 + 3(-14) - 4(5) = 7$$

$$x_2 - 42 - 20 = 7$$

$$x_2 - 62 = 7$$

$$x_2 = 69$$

$$x_1 - 2(69) - 5(-14) + 3(5) = 2$$

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Some systems can't be put into triangular form but something similar echelon form.

A system in echelon form has the following properties

1. Every variable is the leading variable of at most one equation.
2. There are no solutions, exactly one solution or infinitely many solutions.

It is worth noting that triangular systems are also echelon.

Example: Find all solutions to the system of linear equations given by:

$$x_1 + 2x_2 - 5x_3 = -4$$

$$-2x_2 + 4x_3 = 8$$

The first thing to note here is that we have no equation telling us what x_3 is. We can modify our system in the following way.

$$x_1 + 2x_2 - 5x_3 = -4$$

$$-2x_2 + 4x_3 = 8$$

$$0x_3 = 0$$

We call x_3 a **free variable**

$$x_3 = t$$

$$-2x_2 + 4t = 8$$

$$-2x_2 = 8 - 4t$$

$$x_2 = -4 + 2t$$

$$x_1 + 2(-4 + 2t) - 5(t) = -4$$

$$x_1 + -8 + 4t - 5t = -4$$

$$x_1 + -8 - t = -4$$

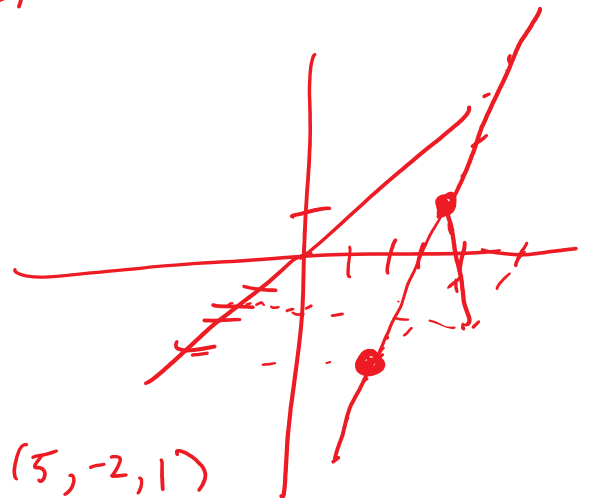
$$x_1 = 4 + t$$

$$x_1 = 4 + t$$

$$x_2 = -4 + 2t$$

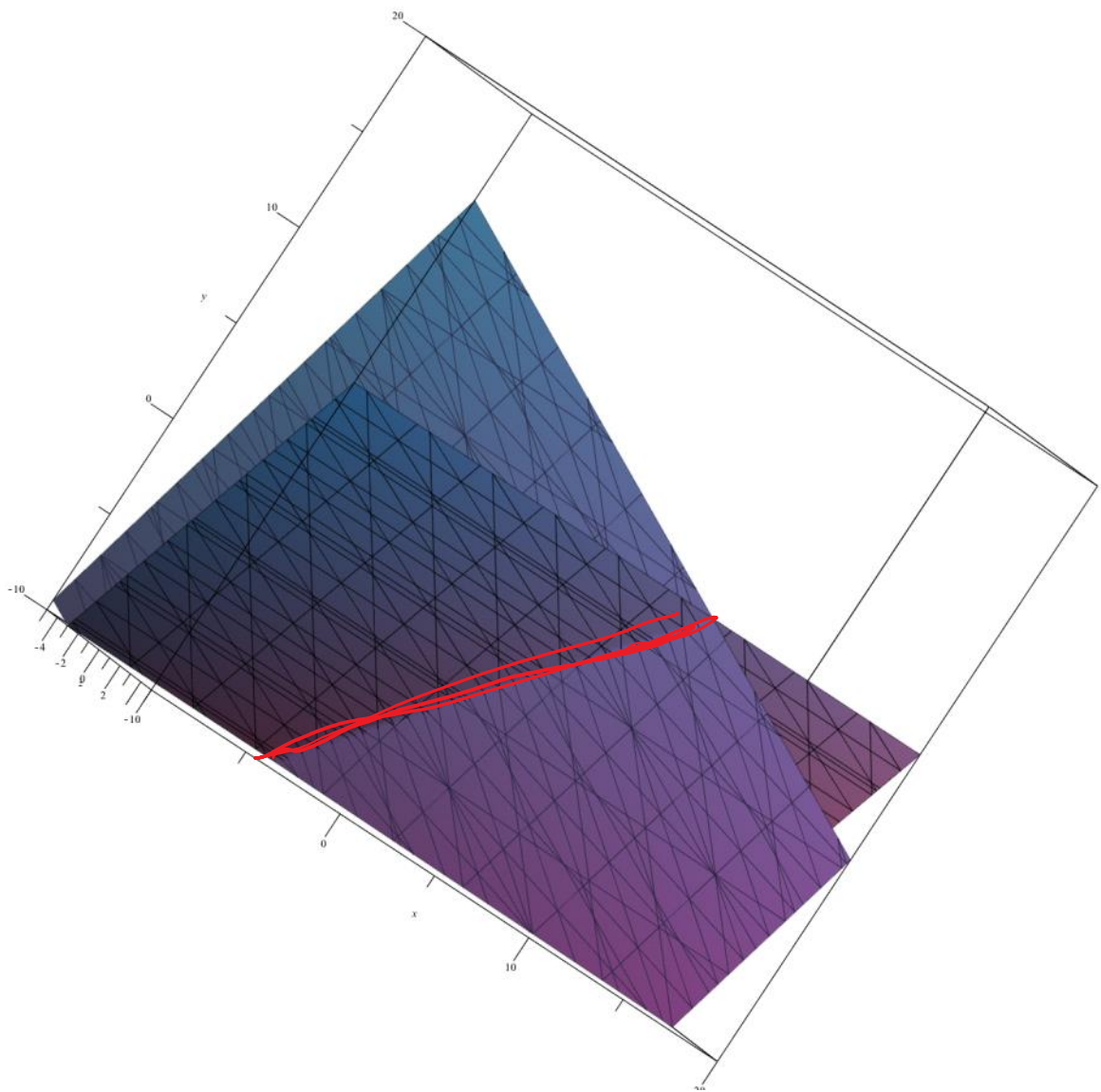
$$x_3 = t$$

$$(4, -4, 0), (5, -2, 1)$$



Section 1.1 Linear Equations

$$\left\{ \begin{array}{l} x_1 + 2x_2 - 5x_3 = -4 \\ -2x_2 + 4x_3 = 8 \\ 0x_3 = 0 \end{array} \right.$$



Section 1.1 Linear Equations

Find all solutions to the system of linear equations

$$n = 5$$
$$m = 3$$

$$\begin{cases} x_1 + 2x_2 - x_3 + 3x_5 = 7 & (1) \\ x_2 - 4x_3 + x_5 = -2 & (2) \\ x_4 - 2x_5 = 1 & (3) \end{cases}$$

$\downarrow$
5

$x_3 = s$
 $x_5 = t$

$$(3) \quad x_4 - 2t = 1$$

$x_4 = 1 + 2t$

$$(2) \quad x_2 - 4s + t = -2$$

$x_2 = -2 + 4s - t$

$$(1) \quad x_1 + 2(-2 + 4s - t) - s + 3t = 7$$

$$\leftarrow x_1 - 4 + 8s - 2t - s + 3t = 7$$

$x_1 = -t - 7s + 11$

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When there are infinitely many solutions

$$2(x_1 + 2x_2 - 5x_3) = -4 \quad | \cdot 2$$

$$-2x_2 + 4x_3 = 8$$

$$-2x_1 - 4x_2 + 10x_3 = 8$$

$$2x_1 + 4x_2 - 10x_3 = -8$$

$$+ -2x_1 - 4x_2 + 10x_3 = 8$$

$$0x_3 = 0$$

When there are no solutions

$$x_1 + 2x_2 - 5x_3 = -4$$

$$-2x_2 + 4x_3 = 8$$

$$-2x_1 - 4x_2 + 10x_3 = 9$$

$$2x_1 + 4x_2 - 10x_3 = -8$$

$$+ -2x_1 - 4x_2 + 10x_3 = 9$$

$$0 = 1$$

When there is one solution

$$x_1 + 2x_2 - 5x_3 = -4$$

$$-2x_2 + 4x_3 = 8$$

$$x_3 = 1$$

$$\cancel{2x_1 - 4x_2 + 12x_3 = 10}$$

$$2x_1 + 4x_2 - 10x_3 = -8 \quad \Rightarrow \quad 2x_3 = 2 \quad x_3 = 1$$

$$+ -2x_1 - 4x_2 + 12x_3 = 10$$

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Homework Example

Determine if the following are solutions to the system

$$3x_1 + 8x_2 - 14x_3 = 8$$

$$x_1 + 3x_2 - 4x_3 = 2$$

(a) $(\underline{3 - 2s_1}, \underline{4 + 3s_1}, \underline{s_1})$

$x_1 \quad x_2 \quad x_3$

(b) $(\underline{-2 - 5s_1}, \underline{s_1}, \underline{-(2 + s_1)/2})$

$$3(3 - 2s_1) + 8(4 + 3s_1) - 14s_1 = 8$$

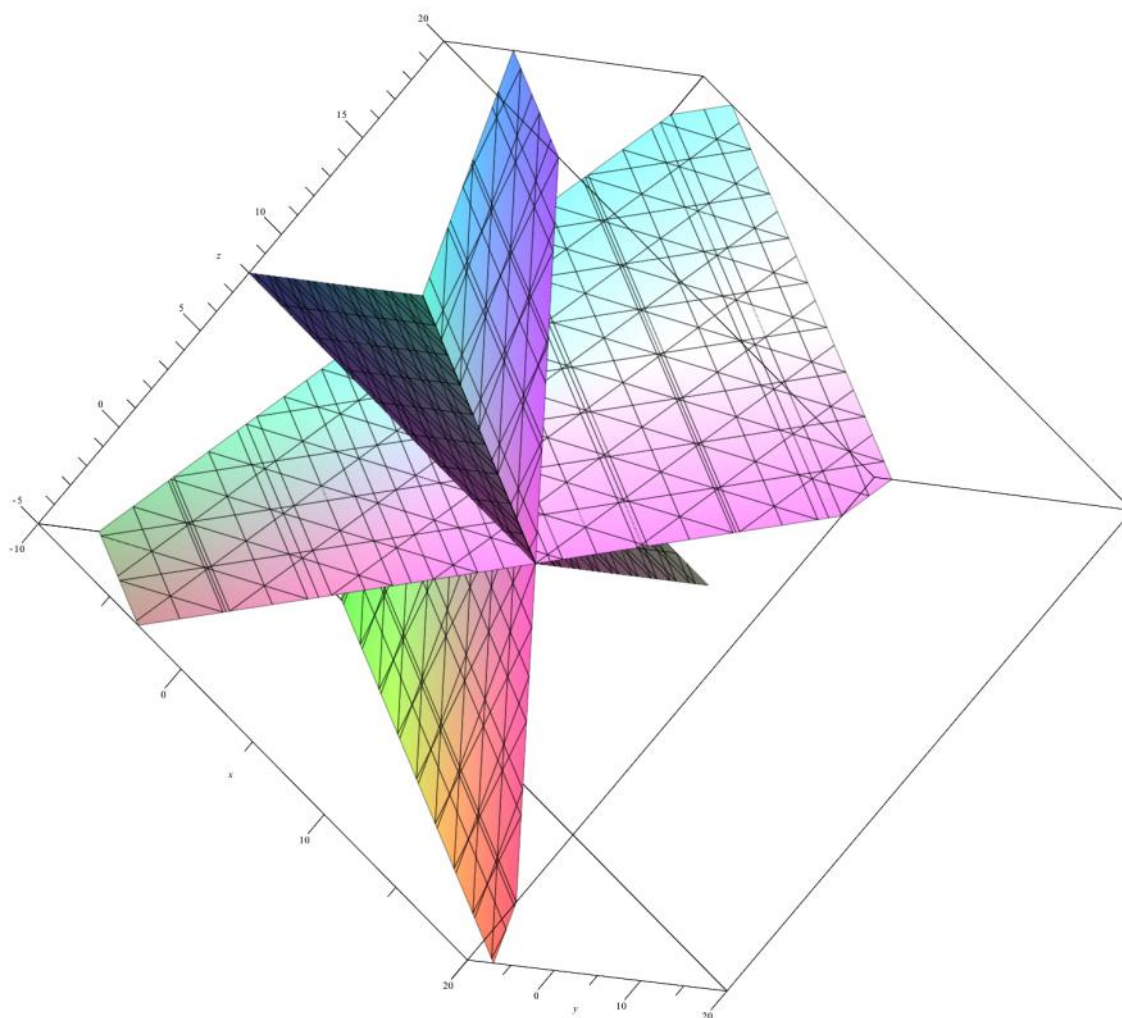
$$\underline{9} - \underline{6s_1} + 32 + \underline{24s_1} - \underline{14s_1} = 8$$

Section 1.1 Linear Equations

$$x + y + z = 8$$

$$-x - y + z = 8$$

$$z = 8$$



Section 1.1 Linear Equations

$$x + y + z = 8$$

$$-x - y + z = 8$$

$$z = 15$$

