

Section 1.2 Linear Systems and Matrices

Systems of linear equations are both mathematically interesting and useful in a wide array of applications. As a result finding ways to solve and manipulate them is worthwhile! In the last section we worked on solving systems in echelon form, however most linear systems don't present themselves in such a canonical way. In this section we will cover Gaussian Elimination, a solution method, and introduce another way to represent a linear system, as an augmented matrix.

Equivalent Linear Systems: Two linear systems are said to be equivalent if they have the same solutions.

Ex. The three following linear systems are equivalent

$$\begin{cases} 2x_1 + x_2 - 4x_3 = -7 \\ 4x_1 + 2x_2 + x_3 = 4 \\ 2x_1 - x_2 + 2x_3 = 7 \end{cases} \rightarrow \begin{cases} 2x_1 + x_2 - 4x_3 = -7 \\ -2x_2 + x_3 = 4 \\ 9x_3 = 18 \end{cases} \rightarrow \begin{cases} x_1 = 1 \\ x_2 = -1 \\ x_3 = 2 \end{cases}$$

$$\begin{aligned} 2(1) + (-1) - 4(2) &= -7 \checkmark \\ 4(1) + 2(-1) + 2 &= 4 \checkmark \\ 2(1) - (-1) + 2(2) &= 7 \checkmark \end{aligned} \quad \begin{aligned} 2(1) + (-1) - 4(2) &= -7 \checkmark \\ -2(-1) + 2 &= 4 \checkmark \\ 9(2) &= 18 \checkmark \end{aligned}$$

The triangular system is the one that is easy to solve, given that they are equivalent we would like to be able to transform one form of the system into the other so that we can obtain the solutions. We do this with **elementary row operations**.

Elementary Operations: Operations on a linear system of equations which do not change the solution set.

1. Interchange any two equations. (*It does not matter where we put them on the page*)
2. Multiply any equation by a nonzero constant. (*We can multiply both sides of an equation and not change its solution*)
3. Add a multiple of one equation to another. (*This is just the algebraic operation of adding the same thing to both sides of an equation*)

Let's just review why (3) works.

any (3) works.

$$(-2)(2x_1 + x_2 - 4x_3) - (-7)(-2)$$
$$4x_1 + 2x_2 + x_3 = 4$$
$$\underline{-4x_1 - 2x_2 + 8x_3 = 14}$$
$$4x_1 + 2x_2 + x_3 + 14 = 4 + 14$$
$$\underline{\cancel{4x_1} + \cancel{2x_2} + \underline{x_3} - \cancel{4x_1} - \cancel{2x_2} + \underline{8x_3} = 18}$$
$$\underline{9x_3 = 18}$$

Section 1.2 Linear Systems and Matrices

Let's transform this into echelon form using elementary operations

$$\begin{aligned} -2(2x_1 + x_2 - 4x_3) &= -7(1) \\ \rightarrow 4x_1 + 2x_2 + x_3 &= 4 \\ 2x_1 - x_2 + 2x_3 &= 7 \end{aligned}$$

$$-2e_1 + e_2$$

$$-4x_1 - 2x_2 + 8x_3 = 14$$

$$+4x_1 + 2x_2 + x_3 = 4$$

$$9x_3 = 18$$

$$-1(2x_1 + x_2 - 4x_3) = -7(-1) \quad -1(e_1) + e_3$$

$$9x_3 = 18$$

$$\rightarrow 2x_1 - x_2 + 2x_3 = 7$$

$$\begin{aligned} -2x_1 - x_2 + 4x_3 &= 7 \\ +2x_1 - x_2 + 2x_3 &= 7 \end{aligned}$$

$$-2x_2 + 6x_3 = 14$$

$$2x_1 + x_2 - 4x_3 = -7$$

$$9x_3 = 18$$

$$-2x_2 + 6x_3 = 14$$

swap

$$2x_1 + x_2 - 4x_3 = -7 \rightarrow 2x_1 + (-1) - 4(2) = -7 \rightarrow 2x_1 - 9 = -7 \rightarrow 2x_1 = 2 \rightarrow x_1 = 1$$

$$-2x_2 + 6x_3 = 14 \rightarrow -2x_2 + 6(2) = 14 \rightarrow -2x_2 + 12 = 14 \rightarrow -2x_2 = 2 \rightarrow x_2 = -1$$

$$9x_3 = 18 \rightarrow x_3 = \frac{18}{9} = 2$$

$$\begin{aligned} x_1 &= 1 \\ x_2 &= -1 \\ x_3 &= 2 \end{aligned}$$

Section 1.2 Linear Systems and Matrices

The Augmented Matrix

Elimination can be tedious and so if we can reduce the monotony a bit we should. One way is stop writing each variable every step. Instead we will use an array of numbers where the position of that number assigns it as a coefficient of a specific variable or constant.

$$\begin{array}{l} 2x_1 + x_2 - 4x_3 = -7 \\ 4x_1 + 2x_2 + x_3 = 4 \\ 2x_1 - x_2 + 2x_3 = 7 \end{array} \xrightarrow{\begin{matrix} e_1 \\ e_2 \\ e_3 \end{matrix}} \begin{bmatrix} 2x_1 & 1x_2 & -4x_3 & -7 \\ 4x_1 & 2x_2 & 1x_3 & 4 \\ 2x_1 & -1x_2 & 2x_3 & 7 \end{bmatrix}$$

columns x_i

Row i

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} & b_{11} \\ a_{21} & a_{22} & \dots & a_{2n} & b_{21} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} & b_{m1} \end{bmatrix}$$

Anatomy of an augmented matrix

With augmented matrices we perform **elementary row operations** just like the elementary operation but on rows in our matrix instead of on equations.

- 1. Interchange any two rows.
- 2. Multiply any row by a nonzero constant.
- 3. Add a multiple of one equation to another.

Gaussian Elimination

$$\begin{bmatrix} 2 & 1 & -4 & -7 \\ 4 & 2 & 1 & 4 \\ 2 & -1 & 2 & 7 \end{bmatrix} \xrightarrow{\text{Row operations to echelon form}} \begin{bmatrix} 2 & 1 & -4 & -7 \\ 0 & -2 & 1 & 4 \\ 0 & 0 & 9 & 18 \end{bmatrix} \xrightarrow{\text{Back to system}} \begin{cases} 2x_1 + x_2 - 4x_3 = -7 \\ -2x_2 + x_3 = 4 \\ 9x_3 = 18 \end{cases} \xrightarrow{\text{Back solve}} \begin{cases} x_1 = 1 \\ x_2 = -1 \\ x_3 = 2 \end{cases}$$

$9x_3 = 18$

Typical Procedure:

1. If it makes the things easier interchange rows (If a row has a 1 in first column make it the first row)
2. Use Row 1 to eliminate $a_{12} \dots a_{1n}$ (get all zeros below a_{11} in the first column)
3. Use Row 2 to eliminate $a_{23} \dots a_{2n}$ (get all zeros below a_{22} in the second column)
4. Continue this approach working from left to right until you are in echlon form.

Let's do this!

$$\begin{bmatrix} 2 & 1 & -4 & -7 \\ 4 & 2 & 1 & 4 \\ 2 & -1 & 2 & 7 \end{bmatrix} \xleftarrow{\begin{matrix} -2R_1 \rightarrow -4 & -2 & 8 & 14 \\ + R_2 & 4 & 2 & 1 & 4 \\ \hline & 0 & 0 & 9 & 18 \end{matrix}}$$

$$\begin{bmatrix} 2 & 1 & -4 & -7 \\ 0 & 0 & 9 & 18 \\ 2 & -1 & 2 & 7 \end{bmatrix} \xleftarrow{\begin{matrix} (-1)R_1 & -2 & -1 & 4 & 7 \\ + R_3 & 2 & -1 & 2 & 7 \\ \hline & 0 & -2 & 6 & 14 \end{matrix}}$$

$$\begin{bmatrix} 2 & 1 & -4 & -7 \\ 0 & 0 & 9 & 18 \\ 0 & -2 & 6 & 14 \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 1 & -4 & -7 \\ 0 & -2 & 6 & 14 \\ 0 & 0 & 9 & 18 \end{bmatrix}$$

Note: In lecture, I thought I made an error here. There is not actually one, we just ended up with a different equivalent form because of the choice we made for row operations.

Section 1.2 Linear Systems and Matrices

We call spots occupying a leading term a **pivot position** and the column it lives in is a **pivot column**

$$\begin{bmatrix} 2 & 1 & -4 & -7 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & 9 & 18 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & -4 & 1 & 0 & -7 \\ 0 & -2 & 1 & 2 & 10 & 4 \\ 0 & 0 & 3 & 0 & 2 & 18 \\ 0 & 0 & 0 & 0 & 5 & 10 \end{bmatrix}$$

With this terminology in mind we can introduce **Reduced Row Echelon form**

A matrix is in reduced row echelon (rref) form if:

1. It is in echelon form.
2. All pivot positions contain a 1.
3. The only nonzero term in a pivot column is in the pivot position.

$$\begin{array}{l} e_1 \\ e_2 \\ e_3 \end{array} \begin{bmatrix} x_1 & x_2 & x_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\textcircled{1} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 2/3 \\ 0 & 1 & 0 & -1 & 0 & 31/3 \\ 0 & 0 & 1 & 0 & 0 & 14/3 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

Let's extract the solutions from these matrices in rref form

$$\begin{aligned} x_1 + 0x_2 + 0x_3 &= 1 \\ 0x_1 + x_2 + 0x_3 &= -1 \\ 0x_1 + 0x_2 + x_3 &= 2 \end{aligned}$$

$$\begin{aligned} x_1 &= 1 \\ x_2 &= -1 \\ x_3 &= 2 \end{aligned}$$

$$\begin{aligned} \textcircled{1} \quad x_1 + 0x_2 + 0x_3 + x_4 + 0x_5 &= 2/3 \\ 0x_1 + x_2 + 0x_3 - x_4 + 0x_5 &= 31/3 \\ 0x_1 + 0x_2 + x_3 + 0x_4 + 0x_5 &= 14/3 \\ 0x_1 + 0x_2 + 0x_3 + 0x_4 + x_5 &= 2 \end{aligned}$$

$$x_5 = 2$$

$$x_3 = 14/3$$

$$x_2 - x_4 = 31/3$$

$$x_1 + x_4 = 2/3$$

$$x_4 = t$$

$$x_2 - t = 31/3$$

$$x_2 = 31/3 + t$$

$$x_1 + t = 2/3$$

$$x_1 = 2/3 - t$$

Section 1.2 Linear Systems and Matrices

Gauss-Jordan Elimination

If we go all the way to rref form we call this Gauss-Jordan Elimination. Let's take our first system to rref form

$$\left[\begin{array}{ccc|c} 2 & 0 & -4 & -7 \\ 0 & -2 & 1 & 4 \\ 0 & 0 & 9 & 18 \end{array} \right] \quad \left\{ \begin{array}{l} \text{Typical procedure: Use the pivot positions to obtain zeros in their columns then divide} \\ \text{the rows by the pivot position values to obtain 1's there (This can be done first as well)} \end{array} \right.$$

$$\begin{array}{rcl} & 0 & -1 \frac{1}{2} \quad 2 \\ \frac{1}{2} R_2 & & & \\ + R_1 & 2 & 1 & -4 \quad -7 \\ \hline & 2 & 0 & -\frac{7}{2} \quad -5 \end{array}$$

$$\left[\begin{array}{ccc|c} 2 & 0 & -\frac{7}{2} & -5 \\ 0 & -2 & 1 & 4 \\ 0 & 0 & 9 & 18 \end{array} \right] \quad \begin{array}{l} \nearrow \\ \frac{9}{9} \end{array} \quad \left[\begin{array}{ccc|c} 0 & 0 & 1 & 2 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 2 & 0 & -\frac{7}{2} & -5 \\ 0 & -2 & 1 & 4 \\ 0 & 0 & 1 & 2 \end{array} \right] \quad \begin{array}{l} -R_3 \\ +R_2 \end{array} \quad \left[\begin{array}{ccc|c} 0 & 0 & -1 & -2 \\ 0 & -2 & 1 & 4 \\ 0 & -2 & 0 & 2 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 2 & 0 & -\frac{7}{2} & -5 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array} \right] \quad \begin{array}{l} \frac{7}{2} R_3 \\ +R_1 \end{array} \quad \left[\begin{array}{ccc|c} 0 & 0 & \frac{7}{2} & 7 \\ 2 & 0 & -\frac{7}{2} & -5 \\ 2 & 0 & 0 & 2 \end{array} \right]$$

$$\rightarrow \left[\begin{array}{ccc|c} 2 & 0 & 0 & 2 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & 1 & 2 \end{array} \right] \quad \begin{array}{l} \frac{1}{2} R_1 \\ -\frac{1}{2} R_2 \end{array} \quad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array} \right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array} \right] \rightarrow \begin{array}{l} x_1 = 1 \\ x_2 = -1 \\ x_3 = 2 \end{array}$$

Section 1.2 Linear Systems and Matrices

Computers can do elimination, however make sure you can too! Chances are an exam or quiz problem will require hand calculations!

Use technology to find the solution to the system

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 1 \\ 4x_1 + 5x_2 + 6x_3 = 2 \\ 7x_1 + 8x_2 + 9x_3 = 3 \end{cases}$$

Mathematica

→ $A = \{\{1, 2, 3, 1\}, \{4, 5, 6, 2\}, \{7, 8, 9, 3\}\}$
 RowReduce[A]
 MatrixForm[%]

To execute the code above hit shift+enter

Untitled-1 - Wolfram Mathematica 11.3

File Edit Insert Format Cell Graphics Evaluation Palettes Window Help

In[8]:= A = {{1, 2, 3, 1}, {4, 5, 6, 2}, {7, 8, 9, 3}}
 RowReduce[A]
 MatrixForm[%]

Out[8]= {{1, 2, 3, 1}, {4, 5, 6, 2}, {7, 8, 9, 3}}

Out[10]= {{1, 0, -1, -1/3}, {0, 1, 2, 2/3}, {0, 0, 0, 0}}

Out[11]=MatrixForm=

$$\begin{pmatrix} 1 & 0 & -1 & -\frac{1}{3} \\ 0 & 1 & 2 & \frac{2}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$0x_4 = 0$
 $x_4 = 0$

Maple

with(linalg)

A:=matrix(3,4,[1,2,3,1,4,5,6,2,7,8,9,3])
 rref(A)

Untitled (1) - [Server 1] - Maple 16

File Edit View Insert Format Table Drawing Plot Spreadsheet Tools Window Help

with(linalg):
 A := matrix(3, 4, [1, 2, 3, 1, 4, 5, 6, 2, 7, 8, 9, 3])

$$\begin{pmatrix} 1 & 2 & 3 & 1 \\ 4 & 5 & 6 & 2 \\ 7 & 8 & 9 & 3 \end{pmatrix}$$

rref(A)

$$\begin{pmatrix} 1 & 0 & -1 & -\frac{1}{3} \\ 0 & 1 & 2 & \frac{2}{3} \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Matlab or Octave (Octave is free!)

A=[1,2,3,1; 4,5,6,2 ; 7,8,9,3]

Rref(A)

You can install octave on your machine or take advantage of

<https://octave-online.net/>

Command Window

>> A=[1,2,3,1;4,5,6,2;7,8,9,3]

A =

1 2 3 1
 4 5 6 2
 7 8 9 3

>> rref(A)

ans =

1.0000 0 -1.0000 -0.3333
 0 1.0000 2.0000 0.6667
 0 0 0 0

octave:2> A=[1,2,3,1;4,5,6,2;7,8,9,3]

A =

1 2 3 1
 4 5 6 2
 7 8 9 3

octave:3> rref(A)

ans =

1.00000 0.00000 -1.00000 -0.33333
 0.00000 1.00000 2.00000 0.66667
 0.00000 0.00000 0.00000 0.00000

Section 1.2 Linear Systems and Matrices

Office Hours Schedule

With	When	Where
Christian	Monday 3:00pm-4:00pm	Phillips 344
Christian	Friday 12:00pm-1:00pm	Phillips 344
Wyatt	Wednesday 2:00pm-3:00pm	Phillips 030
Joseph	Thursday 2:00pm-3:00pm	Phillips 410
Keon	Wednesday 12:10pm-1:10pm	Phillips 403

← probably
will move
to classroom