

Section 10.1 Inner Products

In chapter 8 we saw how useful the dot product between two vectors in R^n was. It allows us to find the angle between vectors, measure how far away two vectors are from each other and create an orthonormal basis from any given basis using the Gram-Schmidt process. As such we would like to extend this idea to other vector spaces and we do through something called the inner product, of which the dot product is an example.

Let $\vec{u}, \vec{v}, \vec{w} \in V$ with V a vector space and let c be a scalar. An inner product on V is a function that takes two vectors as input and produces a scalar as an output. It is often denoted $\langle \vec{u}, \vec{v} \rangle$ and satisfies the following properties.

$$(1) \langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$$

$$(2) \langle \vec{u} + \vec{w}, \vec{v} \rangle = \langle \vec{u}, \vec{v} \rangle + \langle \vec{w}, \vec{v} \rangle$$

$$(3) \langle c\vec{u}, \vec{v} \rangle = \langle \vec{u}, c\vec{v} \rangle = c\langle \vec{u}, \vec{v} \rangle$$

$$(4) \langle \vec{u}, \vec{u} \rangle \geq 0 \text{ and } \langle \vec{u}, \vec{u} \rangle = 0 \text{ only if } \vec{u} = \vec{0}_V$$

Example: Consider the vector space of continuous functions on the interval $[-1, 1]$. Show that the function defined by $\langle f(x), g(x) \rangle = \int_{-1}^1 f(x)g(x)dx$ forms an inner product on $C([-1, 1])$

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There can be more than one inner product on a given space. Consider the space of degree 3 polynomials on $[-1,1]$. Since they are continuous on that interval the inner product we showed before $\langle f(x), g(x) \rangle = \int_{-1}^1 f(x)g(x)dx$ applies to them, so there is one. We could also define a fun innerproduct in the following way.

Let $p, q \in P^3$ and define $\langle p, q \rangle = p(x_1)q(x_1) + p(x_2)q(x_2) + p(x_4)q(x_4) + p(x_4)q(x_4)$ where $\{x_1 \dots x_4\}$ are distinct real numbers. Show that this defines an inner product.

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When we assign an inner product to a vector space we call the pair an inner product space.

Let V be a vector space with inner product $\langle \vec{u}, \vec{v} \rangle$ the pair $(V, \langle \cdot, \cdot \rangle)$ is called an inner product space.

Consider $\mathbb{R}^3$. We can define different innerproducts to it to create different inner product spaces.

Let $V = \mathbb{R}^3$ and $\langle \vec{u}, \vec{v} \rangle_1 = \vec{u} \cdot \vec{v}$

Let $V = \mathbb{R}^3$ and $\langle \vec{u}, \vec{v} \rangle_2 = u_1v_1 + 2u_2v_2 + 3u_3v_3$

Take $\vec{u} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ calculate $\langle \vec{u}, \vec{v} \rangle_1$ and $\langle \vec{u}, \vec{v} \rangle_2$

Two vectors $\vec{u}, \vec{v}$ in an inner product space $(V, \langle \cdot, \cdot \rangle)$ are orthogonal if and only if $\langle \vec{u}, \vec{v} \rangle = 0$

So $\vec{u}, \vec{v}$ above are orthogonal in one inner product space, but not the other!

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Consider the inner product space formed from the vector space of continuous functions on R , $C(R)$, paired with the inner product $\langle f(x), g(x) \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx$ show that $\sin(nx)$, $\sin(mx)$ are orthogonal when $m \neq n$

To do this we will make use of the identity

$$\sin(mx) \sin(nx) = \frac{1}{2} [\cos((m-n)x) - \cos((m+n)x)]$$

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Inner products are nice because they induce a norm on an inner product space.

Let $(V, \langle \cdot, \cdot \rangle)$ be an inner product space and let $\vec{v} \in V$ then the norm of V is $||\vec{v}|| = \sqrt{\langle \vec{v}, \vec{v} \rangle}$

Norms give us ways to measure the distance between two vectors.

Let $\vec{v}_1, \vec{v}_2 \in (V, \langle \cdot, \cdot \rangle)$ the distance between $\vec{v}_1, \vec{v}_2$ is $d(\vec{v}_1, \vec{v}_2) = ||\vec{v}_1 - \vec{v}_2||$

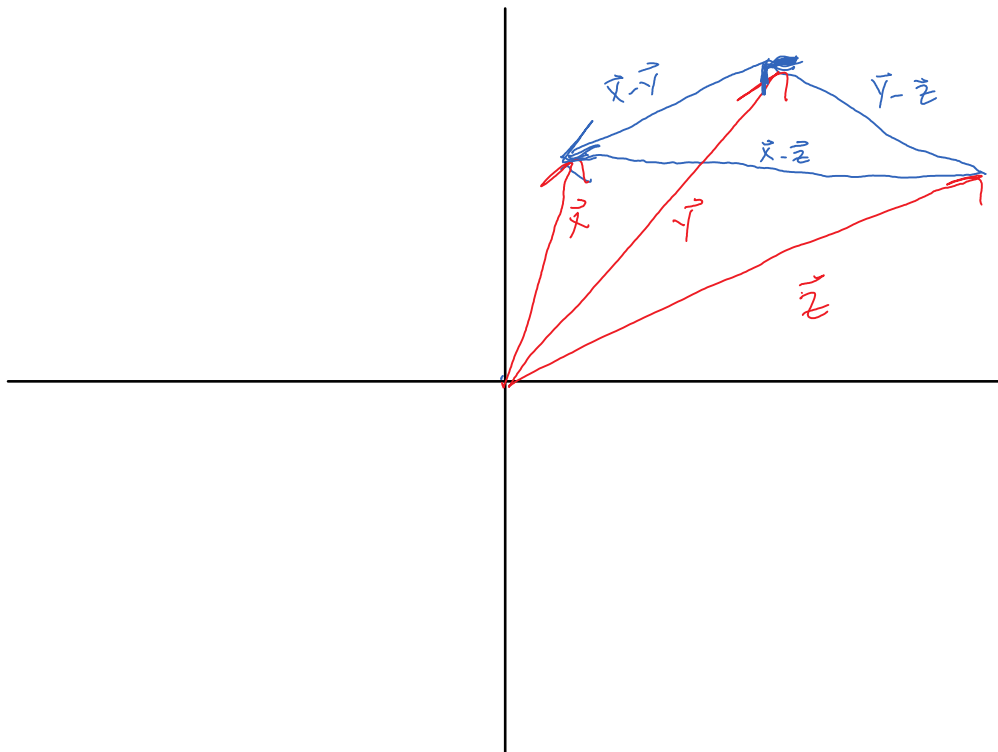
When thought of as a function that measures distance we refer to it as a metric.

Let $d: (x, y) \rightarrow R$ d is a metric if

- (1) $d(x, y) = d(y, x)$
- (2) $d(x, y) \geq 0$
- (3) $d(x, y) = 0$ only if $x = y$
- (4) $d(x, z) \leq d(x, y) + d(y, z)$

A norm, which maps vectors to R , is always a metric, but a metric won't always be a norm. One metric induced by a norm that we have been using is the Euclidean distance.

Property (4) of a metric is just the triangle inequality. If $d(\vec{x}, \vec{y}) = ||\vec{x} - \vec{y}||$. Then $||\vec{x} - \vec{z}|| \leq ||\vec{x} - \vec{y}|| + ||\vec{y} - \vec{z}||$

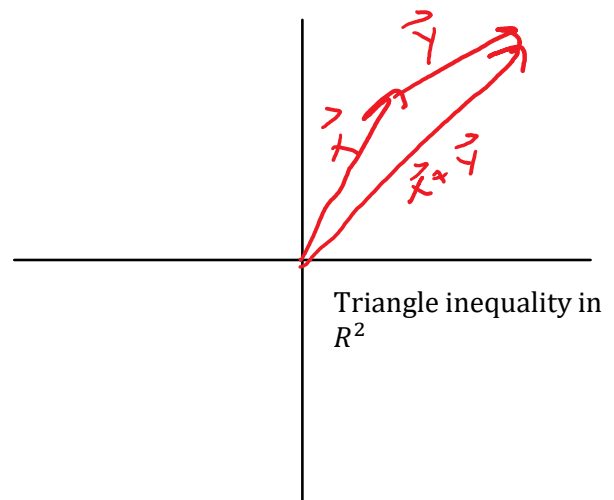


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Another form of the triangle inequality with norms is

$$\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|$$

This is true in any inner product space!



As mentioned before, we can use different norms on the same vector space and which norm you use can have consequences on how the geometry behaves. Let's consider a class of norms called p-norms on R^n .

Let $\vec{x} \in R^n$ and $p \geq 1$ the p -norm of $\vec{x}$ is defined as:

$$\|\vec{x}\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{1/p}$$

To get a sense of how things can differ under different norms, let's consider R^2 with $p = 1, 2, 3, \dots$ and think about what a circle would look like in each innerproduct space. First let's start with $p = 2$, this is our familiar Euclidean distance.

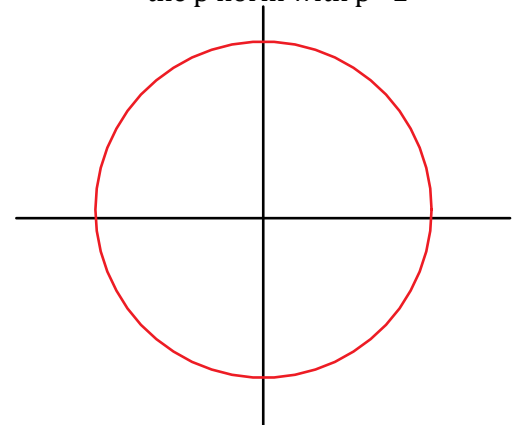
A circle is the set of all points equidistant from a given center. Depending on how we measure distance the shape of this kind of object can change.

Under the $p=2$ norm a circle centered at (x_0, y_0) of radius r would be the set of all points (x, y) such that the length of the vector between them $\begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix}$ would be r , that is the distance between the points is r .

$$\left\| \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} \right\|_2 = r = \sqrt{(x - x_0)^2 + (y - y_0)^2}$$

Or more perhaps more familiar $(x - x_0)^2 + (y - y_0)^2 = r^2$

A circle centered at zero in R^2 with the distance measured from zero using the p norm with $p=2$



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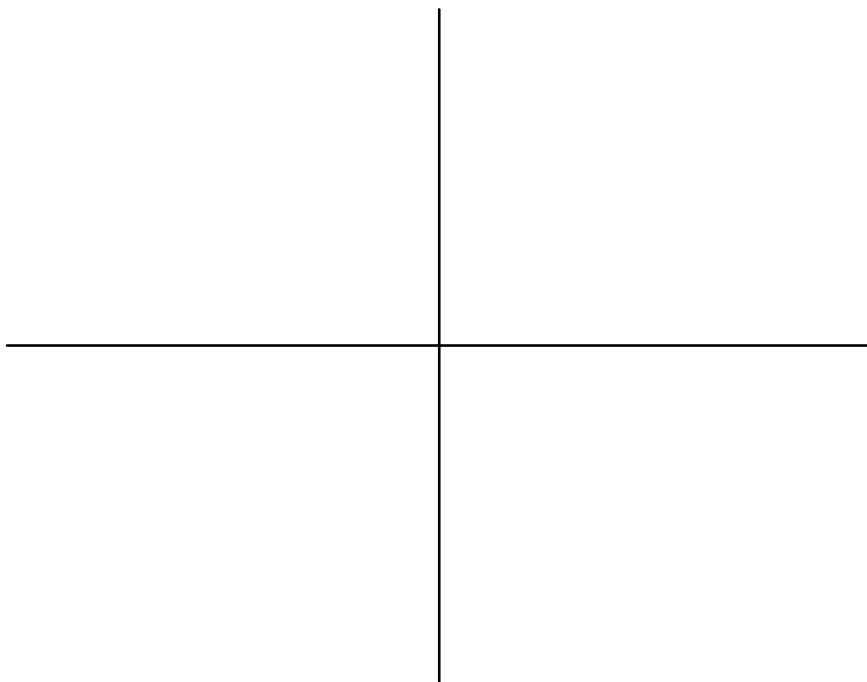
Let's see what this would look like if we take $p = 1$

$$\|\vec{x}\|_p = \left(\sum_{i=1}^n |x_i|^p \right)^{1/p}$$

$$\left\| \begin{bmatrix} x - x_0 \\ y - y_0 \end{bmatrix} \right\|_1 = r = (|x - x_0| + |y - y_0|)^{1/1}$$

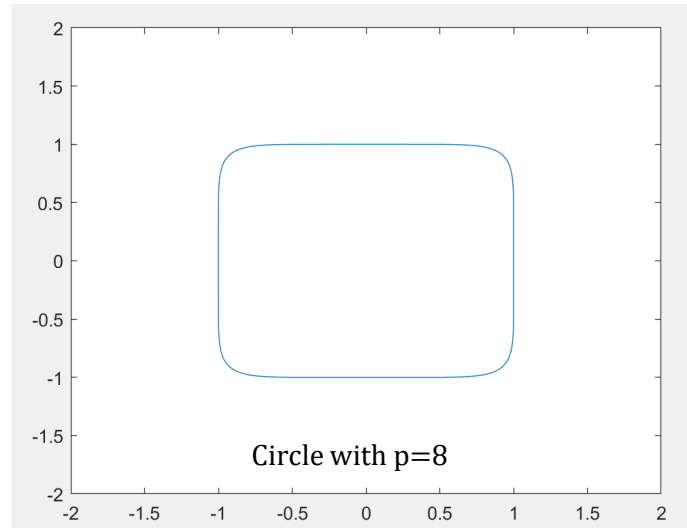
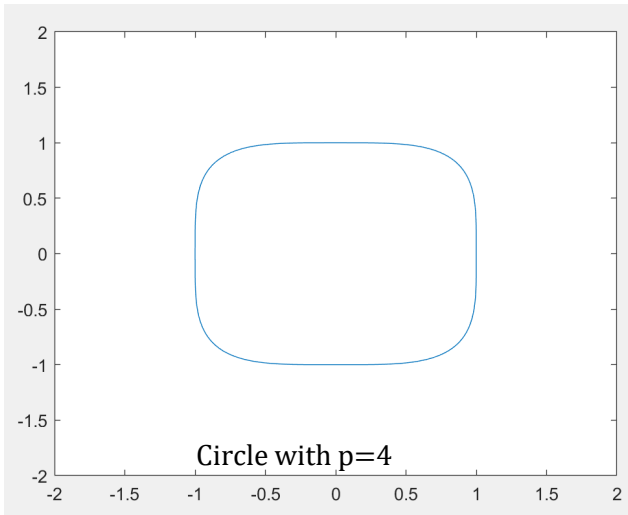
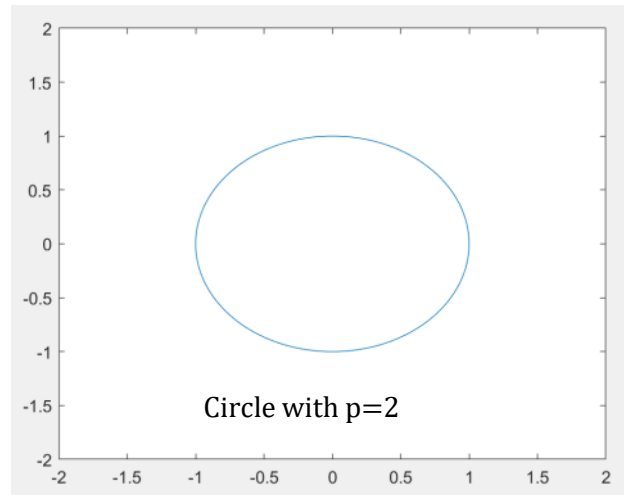
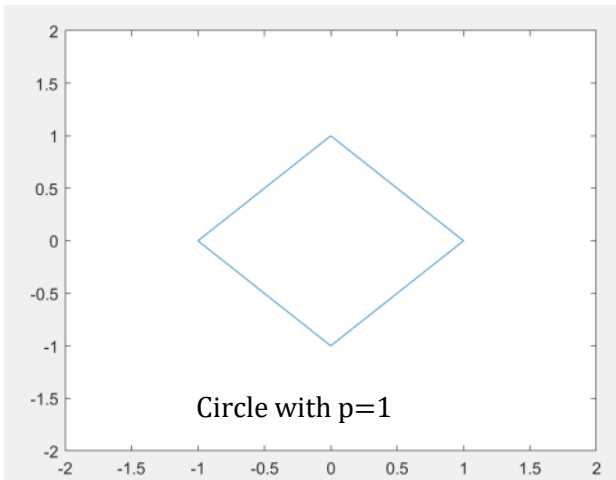
Let's take the center to be (0,0) and radius 1

$$|x| + |y| = 1$$



Let's see what happens as we keep increasing p !

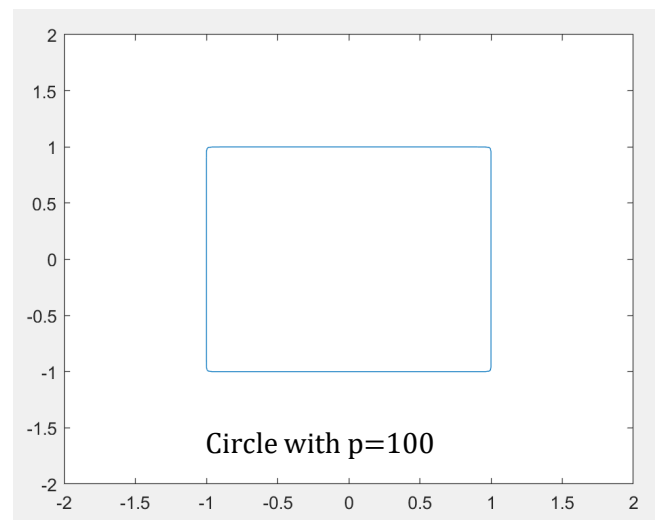
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Some octave code you can play with if you like:
`p=1`
`ezplot(@(x,y) (abs(x).^p +abs( y).^p)^(1/p) - 1)`

Some matlab code(makes better pictures)

```
p=1  
f=@(x,y) (abs(x).^p +abs( y).^p)^(1/p)-1;  
fimplicit(f,[-2 2 -2 2])
```



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We can also do projections in any inner product space.

Let $\vec{u}, \vec{v} \in (V, \langle \cdot, \cdot \rangle)$ with $\vec{v} \neq \vec{0}$. The projection of $\vec{u}$ onto $\vec{v}$ is defined as

$$\text{proj}_{\vec{v}}(\vec{u}) = \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{v}\|^2} \vec{v}$$

Example: Find the projection of $f(x) = x$ onto $g(x) = e^x$ under the norm $\langle f(x), g(x) \rangle = \int_{-1}^1 f(x)g(x)dx$

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Let $\vec{u}, \vec{v} \in (V, \langle \cdot, \cdot \rangle)$ with $\vec{v}$ non zero and let c be a scalar. Then,

- (a) $\text{proj}_{\vec{v}} \vec{u}$ is in the span of $\vec{v}$
- (b) $\vec{u} - \text{proj}_{\vec{v}} \vec{u}$ is orthogonal to $\vec{v}$
- (c) if $\vec{u} \in \text{span}\{\vec{v}\}$, then $\text{proj}_{\vec{v}} \vec{u} = \vec{u}$
- (d) $\text{proj}_{\vec{v}} \vec{u} = \text{proj}_{c\vec{v}} \vec{u}$

What is neat is that this must be true regardless of the inner product.

Note that $\sin(2x)$ is in the span of $\sin(x) \cos(x)$ so the projection of former onto the latter will be itself regardless of the norm.

With $\langle f(x), g(x) \rangle = \int_{-1}^1 f(x)g(x)dx$

$$\text{proj}_{\sin(x) \cos(x)}(\sin(2x)) = \frac{\int_{-1}^1 \sin(2x) \sin(x) \cos(x) dx}{\int_{-1}^1 (\sin(x) \cos(x))^2 dx} \sin(x) \cos(x)$$

Note for any inner product

$$\text{proj}_{\sin(x) \cos(x)}(\sin(2x)) = \frac{\langle \sin(2x), \sin(x) \cos(x) \rangle}{\|\sin(x) \cos(x)\|^2} \sin(x) \cos(x)$$

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The Pythagorean theorem

let $\vec{u}, \vec{v} \in (V, \langle \cdot, \cdot \rangle)$ if $\vec{u} \perp \vec{v}$ then

$$\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$$

Under the inner product $\langle f(x), g(x) \rangle = \int_{-\pi}^{\pi} f(x)g(x)dx$ $\sin(x)$ and $\cos(x)$ are orthogonal.

$$\|\sin(x) + \cos(x)\|^2 = \int_{-\pi}^{\pi} (\sin(x) + \cos(x))^2 dx = \int_{-\pi}^{\pi} \sin^2(x) + \cos^2(x) + 2\sin(x)\cos(x) dx$$

$$\int_{-\pi}^{\pi} \sin^2(x) + \cos^2(x) dx + 2 \int_{-\pi}^{\pi} \sin(x)\cos(x) dx = \int_{-\pi}^{\pi} 1 dx = \pi - (-\pi) = 2\pi$$

$$\|\sin(x)\|^2 + \|\cos(x)\|^2 = \int_{-\pi}^{\pi} \sin^2(x) dx + \int_{-\pi}^{\pi} \cos^2(x) dx = \int_{-\pi}^{\pi} \sin^2(x) + \cos^2(x) dx = 2\pi$$

Cauchy-Schwartz Inequality

$$|\langle \vec{u}, \vec{v} \rangle| \leq \|\vec{u}\| \|\vec{v}\|$$

Proof:

$$\|proj_{\vec{v}}(\vec{u})\| = \left\| \frac{\langle \vec{u}, \vec{v} \rangle}{\|\vec{v}\|^2} \vec{v} \right\| = \frac{|\langle \vec{u}, \vec{v} \rangle|}{\|\vec{v}\|^2} \|\vec{v}\| = \frac{|\langle \vec{u}, \vec{v} \rangle|}{\|\vec{v}\|}$$

$$\text{And } \|proj_{\vec{v}}(\vec{u})\| \leq \|\vec{u}\|$$

So

$$\frac{|\langle \vec{u}, \vec{v} \rangle|}{\|\vec{v}\|} \leq \|\vec{u}\| \rightarrow |\langle \vec{u}, \vec{v} \rangle| \leq \|\vec{u}\| \|\vec{v}\|$$

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