

Section 10.2 Gramm-Schmidt

When we were living in R^n , we found out that orthogonal basis were preferable since they allow us to easily find the coefficients needed to represent any given vector in the space as a linear combination of the basis vectors using the dot product. That is given an orthogonal basis $S = \{\vec{v}_1 \dots \vec{v}_k\}$ and any vector $\vec{v}$ in the span of the basis vectors we could write: $\vec{v} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_k \vec{v}_k$ with $c_1 = \frac{\vec{v}_1 \cdot \vec{v}}{\vec{v}_1 \cdot \vec{v}_1}$, $c_2 = \frac{\vec{v}_2 \cdot \vec{v}}{\vec{v}_2 \cdot \vec{v}_2}$, $\dots$, $c_k = \frac{\vec{v}_k \cdot \vec{v}}{\vec{v}_k \cdot \vec{v}_k}$. When $\vec{v}$ was not in the span of the basis vectors we were able to find the vector that was closest to $\vec{v}$ that is in the span by projecting $\vec{v}$ onto the basis $\hat{v} = \text{proj}_S \vec{v} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_k \vec{v}_k$. This proved very useful as it allowed us to solve the least squares problem, giving us a tool to fit functions to observed data. Recalling that the dot product was an inner product on R^n , we might wonder what we can do with orthogonal bases on more general innerproduct spaces.

First we should define an orthogonal set of vectors.

A set of vectors $\{\vec{v}_1 \dots \vec{v}_k\}$ in an inner product space $(V, \langle \cdot, \cdot \rangle)$ are orthogonal set if $\langle \vec{v}_i, \vec{v}_j \rangle = 0$ for $i \neq j$

Example: $\{1, \sin(x), \cos(x)\}$ form an orthogonal set with respect to $\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)dx$

$$\begin{aligned} \langle 1, \sin(x) \rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} 1 \sin(x) dx = \frac{1}{\pi} [-\cos(x)] \Big|_{-\pi}^{\pi} = \frac{1}{\pi} [-\cos(\pi) - (-\cos(-\pi))] \\ &= \frac{1}{\pi} [-(-1) + -1] = \frac{1}{\pi} [1 - 1] = 0 \end{aligned}$$

$$\langle 1, \cos(x) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} 1 \cos(x) dx = \frac{1}{\pi} \sin(x) \Big|_{-\pi}^{\pi} = \frac{1}{\pi} [\sin(\pi) - \sin(-\pi)] = 0$$

$$\begin{aligned} \langle \sin(x), \cos(x) \rangle &= \frac{1}{\pi} \int_{-\pi}^{\pi} \sin(x) \cos(x) dx = \\ &= \frac{1}{\pi} \int u \cos(x) \frac{du}{\cos(x)} = \frac{1}{\pi} \int u du = \frac{1}{\pi} \frac{u^2}{2} = \frac{1}{\pi} \frac{\sin^2(x)}{2} \Big|_{-\pi}^{\pi} = \frac{1}{\pi} \left[\frac{\sin^2(\pi)}{2} - \frac{\sin^2(-\pi)}{2} \right] = 0 \end{aligned}$$

$u = \sin(x)$
 $\frac{du}{dx} = \cos(x)$
 $dx = \frac{du}{\cos(x)}$

If a set of vectors $\{\vec{v}_1 \dots \vec{v}_k\}$ in an inner product space $(V, \langle \cdot, \cdot \rangle)$ are orthogonal they are linearly independent.

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Let $S = \{\vec{v}_1 \dots \vec{v}_k\}$ be an orthogonal basis for an inner product space $(V, \langle \cdot, \cdot \rangle)$ then for any $\vec{v} \in V$ we can write

$$\vec{v} = s_1 \vec{v}_1 + s_2 \vec{v}_2 + \dots + s_k \vec{v}_k$$

Where $s_i = \frac{\langle \vec{v}_i, \vec{v} \rangle}{\langle \vec{v}_i, \vec{v}_i \rangle}$ for $i = 1 \dots k$

$$\sin^2(x) = \frac{1}{2} - \frac{\cos(2x)}{2}$$

$\vec{v}_1, \vec{v}_2, \vec{v}_3$
 $\{1, \sin(x), \cos(x)\}$ form an orthogonal set with respect to $\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)dx$ express $\sin^2\left(\frac{x}{2}\right)$ as a linear combination of the basis vectors.

$$\sin^2\left(\frac{x}{2}\right) = \frac{1}{2} + 0(\sin(x)) - 1(\cos(x))$$

$$\sin^2\left(\frac{x}{2}\right) = c_1(1) + c_2(\sin(x)) + c_3(\cos(x))$$

$$c_1 = \frac{\langle 1, \sin^2\left(\frac{x}{2}\right) \rangle}{\langle 1, 1 \rangle} = \frac{\frac{1}{\pi} \int_{-\pi}^{\pi} (1 \sin^2\left(\frac{x}{2}\right)) dx}{\frac{1}{\pi} \int_{-\pi}^{\pi} (1) dx} = \frac{\frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2} - \frac{\cos(x)}{2} dx}{\frac{1}{\pi} x \Big|_{-\pi}^{\pi}}$$

$$\boxed{\sin^2\left(\frac{x}{2}\right) = \frac{1}{2} - \cos(x)}$$

$$= \frac{\frac{1}{\pi} \left[\int_{-\pi}^{\pi} \frac{1}{2} - \frac{1}{2} \cos(x) dx \right]}{\frac{1}{\pi} [\pi - (-\pi)]} = \frac{\frac{1}{\pi} \frac{1}{2} x \Big|_{-\pi}^{\pi} - \frac{1}{2\pi} \sin(x) \Big|_{-\pi}^{\pi}}{\frac{2\pi}{\pi}} = \frac{\frac{1}{2\pi} 2\pi}{2} = \boxed{\frac{1}{2}} = c_1$$

$$c_2 = \frac{\langle \sin(x), \sin^2\left(\frac{x}{2}\right) \rangle}{\langle \sin(x), \sin(x) \rangle} = \frac{\frac{1}{\pi} \int_{-\pi}^{\pi} \sin(x) \sin^2\left(\frac{x}{2}\right) dx}{\frac{1}{\pi} \int_{-\pi}^{\pi} \sin^2(x) dx} = \frac{\frac{1}{\pi} \int_{-\pi}^{\pi} \sin(x) \left(\frac{1}{2} - \frac{\cos(x)}{2} \right) dx}{\frac{1}{\pi} \int_{-\pi}^{\pi} \frac{1}{2} - \cos\left(\frac{x}{2}\right) dx}$$

$$= \frac{\frac{1}{2\pi} \int_{-\pi}^{\pi} \sin(x) - \frac{1}{2} \sin(x) \cos(x) dx}{\frac{1}{\pi} \left[\int_{-\pi}^{\pi} \frac{1}{2} - \int_{-\pi}^{\pi} \cos\left(\frac{x}{2}\right) dx \right]} = \frac{0}{\neq} = 0 = c_2$$

$$c_3 = \frac{\langle \cos(x), \sin^2\left(\frac{x}{2}\right) \rangle}{\langle \cos(x), \cos(x) \rangle} = \frac{\frac{1}{\pi} \int_{-\pi}^{\pi} \cos(x) \sin^2\left(\frac{x}{2}\right) dx}{\frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2(x) dx} =$$

$$= \frac{\frac{1}{\pi} \left[\int_{-\pi}^{\pi} \frac{1}{2} \cos(x) - \int_{-\pi}^{\pi} \cos(x) \cos(x) dx \right]}{\frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2(x)} = \frac{-\frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2(x) dx}{\frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2(x) dx} = -1 = c_3$$

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Let S be a subspace of an inner product space $(V, \langle \cdot, \cdot \rangle)$ and suppose that S has an orthogonal basis $\{\vec{v}_1 \dots \vec{v}_k\}$. Then the projection of $\vec{v}$ onto S is given by

$$\text{Proj}_S(\vec{v}) = \frac{\langle \vec{v}_1, \vec{v} \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} \vec{v}_1 + \frac{\langle \vec{v}_2, \vec{v} \rangle}{\langle \vec{v}_2, \vec{v}_2 \rangle} \vec{v}_2 + \dots + \frac{\langle \vec{v}_k, \vec{v} \rangle}{\langle \vec{v}_k, \vec{v}_k \rangle} \vec{v}_k$$

Let S be a non zero subspace of an inner product space with orthogonal basis $\{\vec{v}_1 \dots \vec{v}_k\}$, and let $\vec{u} \in \mathbb{R}^n$. Then,

- (a) $\text{proj}_S \vec{u}$ is in S .
- (b) $\vec{u} - \text{proj}_S \vec{u}$ is orthogonal to S .
- (c) if $\vec{u}$ is in S then $\text{proj}_S \vec{u} = \vec{u}$.
- (d) $\text{proj}_S \vec{u}$ is independent of the choice of orthogonal basis for S .

Find the projection of $f(x) = x^2$ onto $\{1, \sin(x), \cos(x)\}$ with respect to $\langle f, g \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)g(x)dx$

$$\text{Proj}_S \langle x^2 \rangle = c_1 \langle 1 \rangle + c_2 \langle \sin(x) \rangle + c_3 \langle \cos(x) \rangle$$

$$c_1 = \frac{\langle x^2, 1 \rangle}{\langle 1, 1 \rangle} = \frac{\frac{1}{\pi} \int_{-\pi}^{\pi} x^2 dx}{\frac{1}{\pi} \int_{-\pi}^{\pi} 1 dx} = \frac{\frac{1}{\pi} \left[\frac{x^3}{3} \right]_{-\pi}^{\pi}}{\frac{1}{\pi} \left[x \right]_{-\pi}^{\pi}} = \frac{\frac{1}{\pi} \left(\frac{\pi^3}{3} - \frac{-\pi^3}{3} \right)}{\frac{1}{\pi} (\pi - (-\pi))} = \frac{\frac{1}{\pi} \left(\frac{2\pi^3}{3} \right)}{\frac{1}{\pi} (2\pi)} = \frac{\frac{2\pi^3}{3}}{2\pi} = \frac{\pi^2}{3}$$

$$c_2 = \frac{\langle x^2, \sin(x) \rangle}{\langle \sin(x), \sin(x) \rangle} = \frac{\int_{-\pi}^{\pi} x^2 \sin(x) dx}{\int_{-\pi}^{\pi} \sin^2(x) dx}$$

$u = x^2 \quad du = 2x dx$
 $v = \sin(x) \quad dv = \cos(x) dx$
 $\int -x \cos(x) dx$

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$$c_3 = \frac{\langle x^2, \cos(x) \rangle}{\langle \cos(x), \cos(x) \rangle} = \frac{\frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \cos(x) dx}{\frac{1}{\pi} \int_{-\pi}^{\pi} \cos^2(x) dx} = \cancel{\pi} = c_3$$

$$\text{proj}_S(x^2) = c_1(1) + c_2(\sin(x)) + c_3(\cos(x))$$



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We will study some applications of these ideas in the next section, but I promise that these orthogonal bases are very very useful. Given that they are useful we might want to be able to find som. So we return back to Gram-Schmidt!

Gram-Schmidt process. Let $\{\vec{s}_1 \dots \vec{s}_k\}$ be a basis for a subspace S of an inner product space $(V, \langle \cdot, \cdot \rangle)$. An orthogonal basis, $\{\vec{v}_1 \dots \vec{v}_k\}$, can be found for S and is calculated as follows.

$$\begin{aligned}\vec{v}_1 &= \vec{s}_1 \\ \vec{v}_2 &= \vec{s}_2 - \text{proj}_{\vec{v}_1}(\vec{s}_2) \\ \vec{v}_3 &= \vec{s}_3 - \text{proj}_{\vec{v}_1}(\vec{s}_3) - \text{proj}_{\vec{v}_2}(\vec{s}_3) \\ \vec{v}_4 &= \vec{s}_4 - \text{proj}_{\vec{v}_1}(\vec{s}_4) - \text{proj}_{\vec{v}_2}(\vec{s}_4) - \text{proj}_{\vec{v}_3}(\vec{s}_4) \\ &\vdots \\ \vec{v}_k &= \vec{s}_k - \text{proj}_{\vec{v}_1}(\vec{s}_k) - \text{proj}_{\vec{v}_2}(\vec{s}_k) - \text{proj}_{\vec{v}_3}(\vec{s}_k) - \text{proj}_{\vec{v}_4}(\vec{s}_k) - \dots - \text{proj}_{\vec{v}_{k-1}}(\vec{s}_k)\end{aligned}$$

This is the same as our process before, we just use different vector spaces and different inner products for the projections!

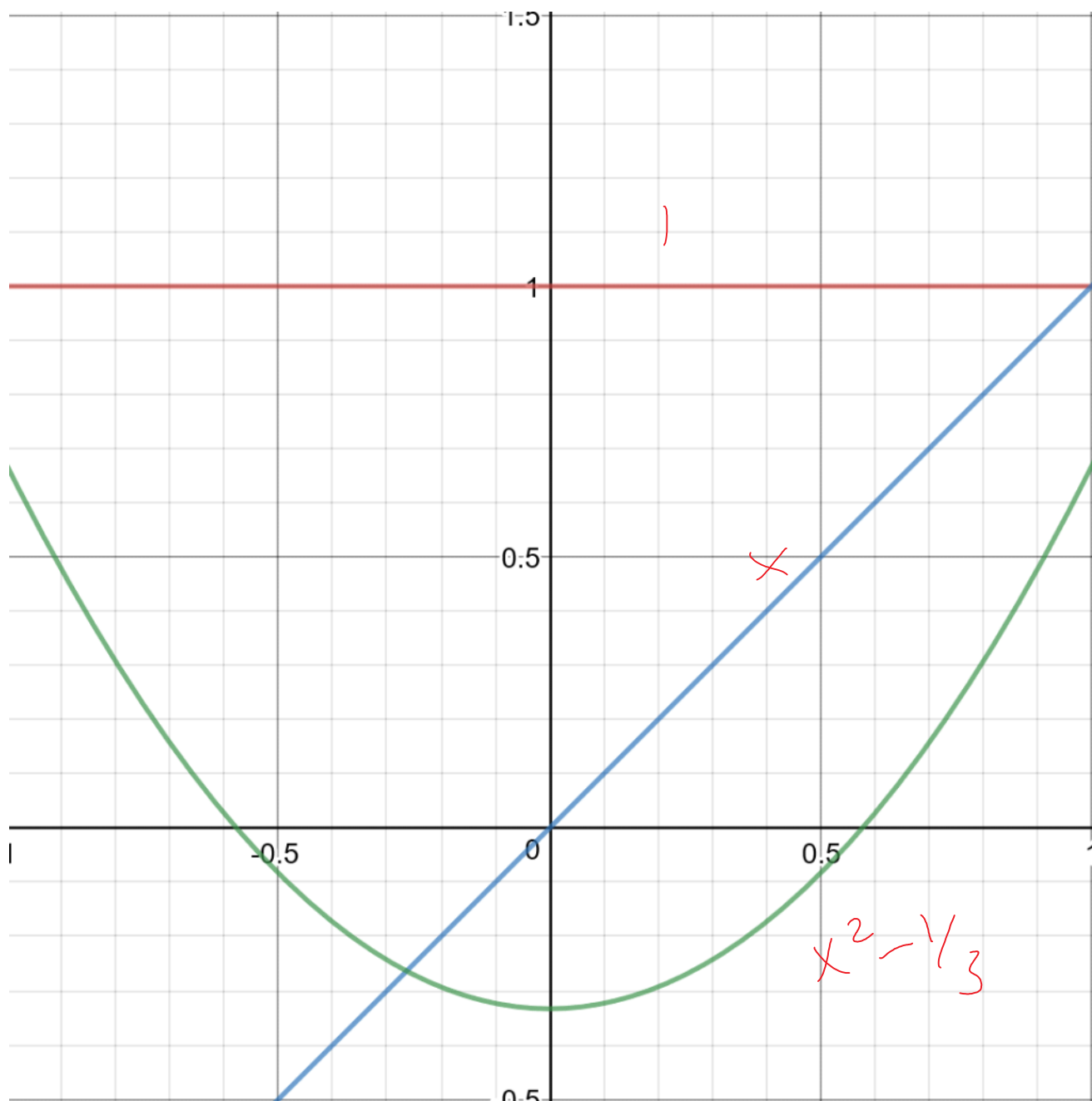
Consider P^2 on $[-1, 1]$ together with $\langle p, q \rangle = \int_{-1}^1 p(x)q(x)dx$ A basis for this space is $\{1, x, x^2\}$
Find an orthogonal basis for this set.

$\begin{matrix} \uparrow & \uparrow & \uparrow \\ S_1 & S_2 & S_3 \end{matrix}$

$$\begin{aligned}\vec{v}_1 &= \vec{s}_1 & \vec{v}_1 &= 1 & \frac{\langle 1, x \rangle}{\langle 1, 1 \rangle} (1) \\ \vec{v}_2 &= x - \text{proj}_{\vec{v}_1}(x) = x - \frac{\int_{-1}^1 (1)x dx}{\int_{-1}^1 (1) dx} (1) = x - \frac{x^2/2 \Big|_{-1}^1}{x \Big|_{-1}^1} (1) = x - \frac{\frac{1}{2} - (-\frac{1}{2})}{1 - (-1)} = 0 \\ &= x \\ \vec{v}_3 &= x^2 - \text{proj}_{\vec{v}_1}(x^2) - \text{proj}_{\vec{v}_2}(x^2) \\ &= x^2 - \frac{\int_{-1}^1 (1)x^2 dx}{\int_{-1}^1 1 dx} - \frac{\int_{-1}^1 x x^2 dx}{\int_{-1}^1 x x dx} = x^2 - \frac{x^3/3 \Big|_{-1}^1}{x \Big|_{-1}^1} - \frac{x^4/4 \Big|_{-1}^1}{x^2/3 \Big|_{-1}^1} \\ &= x^2 - \frac{\frac{1}{3} - (-\frac{1}{3})}{2} - \frac{\frac{1}{4} - (-\frac{1}{4})}{2} = x^2 - \frac{2/3}{2} = x^2 - \frac{1}{3} \\ \vec{v}_1, \vec{v}_2, \vec{v}_3 &= \{1, x, x^2 - \frac{1}{3}\}\end{aligned}$$

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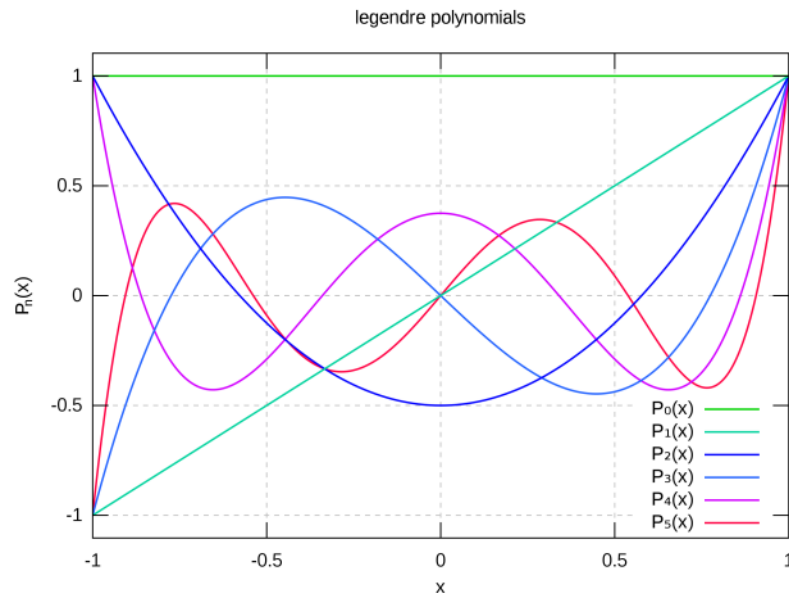
Example Cont.



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One very useful set of orthogonal basis functions for polynomials are the Legendre polynomials.

Legendre polynomials



The first few Legendre polynomials are:

n	$P_n(x)$
0	1
1	x
2	$\frac{1}{2} (3x^2 - 1)$
3	$\frac{1}{2} (5x^3 - 3x)$
4	$\frac{1}{8} (35x^4 - 30x^2 + 3)$
5	$\frac{1}{8} (63x^5 - 70x^3 + 15x)$
6	$\frac{1}{16} (231x^6 - 315x^4 + 105x^2 - 5)$
7	$\frac{1}{16} (429x^7 - 693x^5 + 315x^3 - 35x)$
8	$\frac{1}{128} (6435x^8 - 12012x^6 + 6930x^4 - 1260x^2 + 35)$
9	$\frac{1}{128} (12155x^9 - 25740x^7 + 18018x^5 - 4620x^3 + 315x)$
10	$\frac{1}{256} (46189x^{10} - 109395x^8 + 90090x^6 - 30030x^4 + 3465x^2 - 63)$

These polynomials help us solve Laplace's equation in spherical coordinates

$$\nabla^2 u(\rho, \theta, \phi) = \frac{1}{\rho^2} \frac{\partial}{\partial \rho} \left(\rho^2 \frac{\partial u}{\partial \rho} \right) + \frac{1}{\rho^2 \sin(\phi)} \frac{\partial}{\partial \phi} \left(\sin(\phi) \frac{\partial u}{\partial \phi} \right) + \frac{1}{\rho^2 \sin^2 \phi} \frac{\partial^2 u}{\partial \theta^2} = 0$$

And are often referred to as spherical harmonics.

Laplace's equation is related to many physical phenomena typically in an equilibrium state.

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One of the most widely used sets of orthogonal functions are actually just a bunch of sines and cosines.

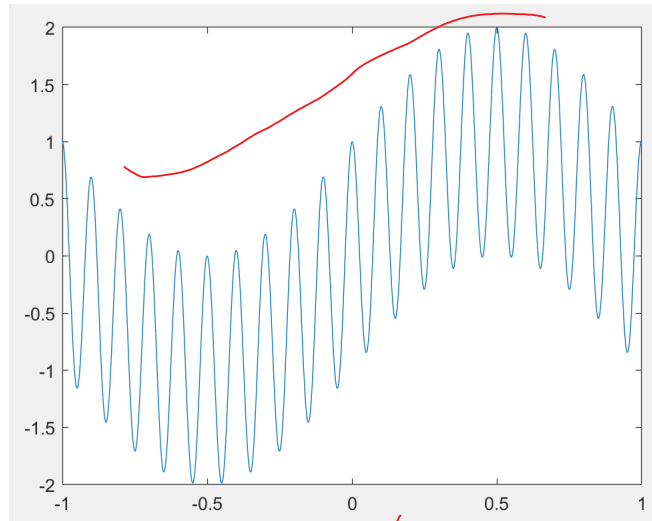
$$\left\{ 1, \cos\left(\frac{\pi x}{L}\right), \sin\left(\frac{\pi x}{L}\right), \cos\left(\frac{2\pi x}{L}\right), \sin\left(\frac{2\pi x}{L}\right), \dots, \cos\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right), \dots \right\}$$

$$\frac{1}{L} \int_{-L}^L f(x) g(x) dx$$

$\langle f, g \rangle$

With these functions we can approximate any function on an interval from $[-L, L]$ using a sum of sines and cosines. That approximation is called the Fourier series and helps us solve PDE's or process signals. The uses of the Fourier series are too numerous to list! One great use is that of filtering out noise from a signal. The Fourier series decomposes a signal into it's various frequencies. If one has high frequency noise in a signal one can remove it

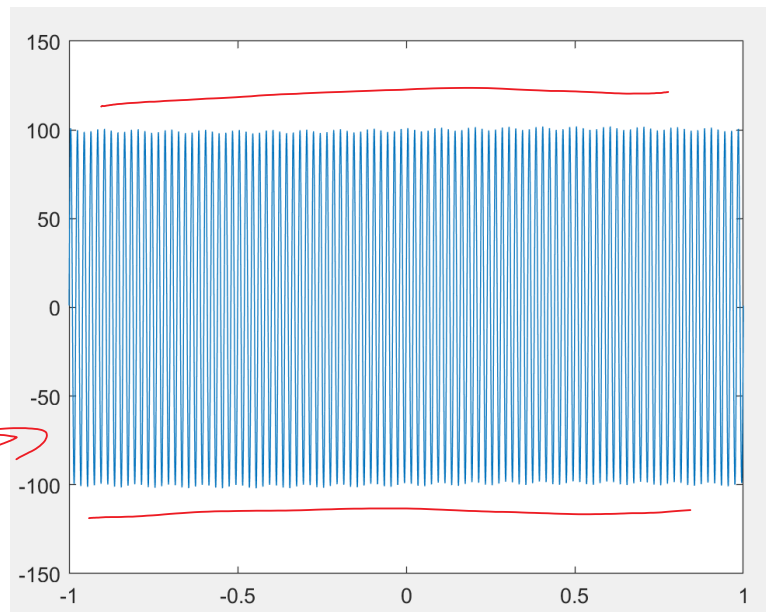
Let's say you expect to get this signal



But what you measure is this signal

Note the extremely high amplitude noise!

IF we project this noise onto the set above, taking enough values of n the signal would decompose into a sum of sines and cosines, one of which would be a high frequency high amplitude function, we could then just chuck it!



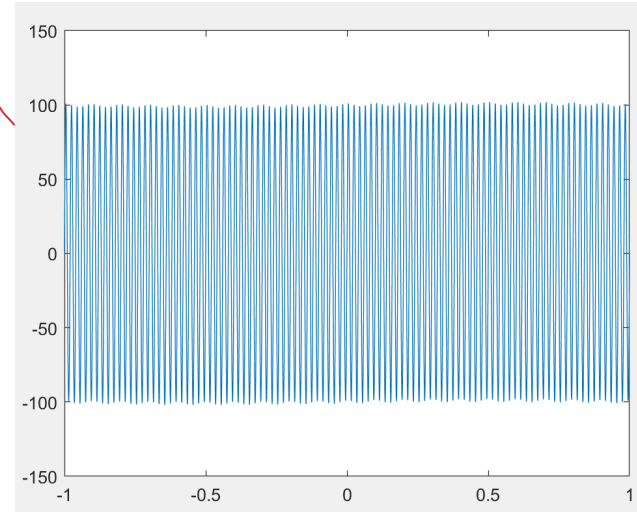
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If we project this signal to the set

$S = \{ \sin(\pi x), \cos(\pi x), \sin(2\pi x), \cos(2\pi x), \dots, \sin(n\pi x), \cos(n\pi x), \dots \}$

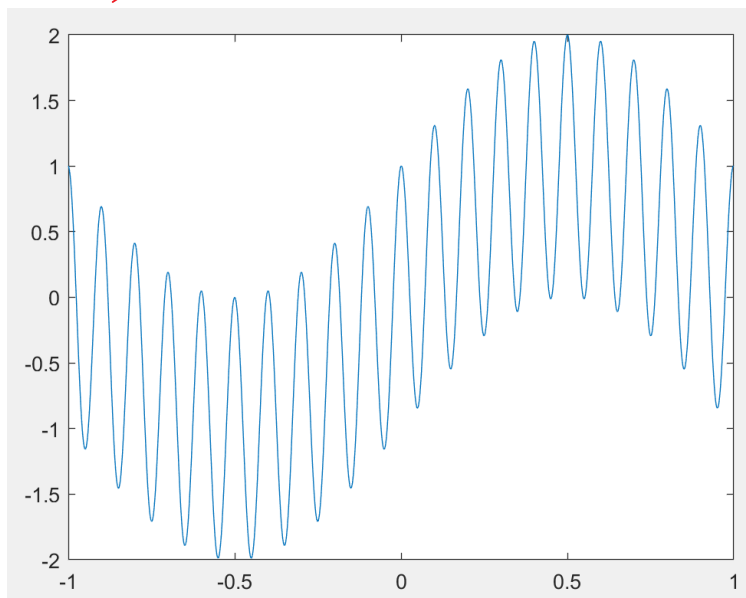
Taking n to 1000 say, we would get .

Proj_S (Signal) $\downarrow$
 $f(x) = \sin(\pi x) + \cos(20\pi x) + 100\sin(100\pi x)$



Dropping the high amplitude high frequency term we get:

$f(x) = \sin(\pi x) + \cos(20\pi x)$



The Fourier series and Fourier transform show up all over in mathematics, especially applied math. To get a sense for this stuff I would recommend watching the video here:

<https://www.youtube.com/watch?v=spUNpyF58BY>

Next class we will talk about the Fourier Series and how it helps us solve PDE's