

Section 10.3 Applications of Inner Products

Let S be a subspace of an inner product space $(V, \langle \cdot, \cdot \rangle)$ and suppose that S has an orthogonal basis $\{\vec{v}_1 \dots \vec{v}_k\}$. Then the projection of $\vec{v}$ onto S is given by

$$\text{Proj}_S(\vec{v}) = \frac{\langle \vec{v}_1, \vec{v} \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} \vec{v}_1 + \frac{\langle \vec{v}_2, \vec{v} \rangle}{\langle \vec{v}_2, \vec{v}_2 \rangle} \vec{v}_2 + \dots + \frac{\langle \vec{v}_k, \vec{v} \rangle}{\langle \vec{v}_k, \vec{v}_k \rangle} \vec{v}_k$$

As it turns out the subspace of functions on the vector space of continuous functions on $[-L, L]$ given by $\text{span}\left\{1, \cos\left(\frac{\pi x}{L}\right), \sin\left(\frac{\pi x}{L}\right), \cos\left(\frac{2\pi x}{L}\right), \sin\left(\frac{2\pi x}{L}\right), \dots, \cos\left(\frac{k\pi x}{L}\right), \sin\left(\frac{k\pi x}{L}\right)\right\}$ is of great use to us. This is because the functions above are orthogonal to each other under the inner product $\langle f, g \rangle = \int_{-L}^L f(x)g(x)dx$.

Let's find the projection of the function $f(x) = -x^2 + 1$ on the subspace we would get if $L = 1$

$$\langle f, g \rangle = \int_{-1}^1 f(x)g(x)dx \quad [-1, 1]$$

$$\text{Proj}_S(-x^2 + 1) = c_0(1) + a_1 \cos(\pi x) + b_1 \sin(\pi x) + \dots + a_k \cos(k\pi x) + b_k \sin(k\pi x)$$

$$\text{Here } c_0 = \frac{\langle 1, -x^2 + 1 \rangle}{\langle 1, 1 \rangle}$$

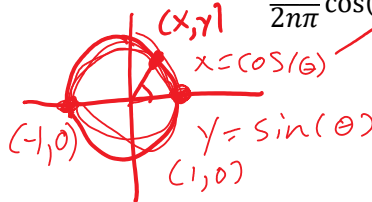
$$a_n = \frac{\langle \cos(n\pi x), -x^2 + 1 \rangle}{\langle \cos(n\pi x), \cos(n\pi x) \rangle}$$

$$b_n = \frac{\langle \sin(n\pi x), -x^2 + 1 \rangle}{\langle \sin(n\pi x), \sin(n\pi x) \rangle} \quad n = 1, 2, \dots, k$$

$$c_0 = \frac{\int_{-1}^1 (-x^2 + 1) dx}{\int_{-1}^1 dx} = \frac{-\frac{x^3}{3} + x \Big|_{-1}^1}{x \Big|_{-1}^1} = \frac{\frac{4}{3}}{2} = \frac{2}{3}$$

$$a_n = \frac{\int_{-1}^1 (-x^2 + 1) \cos(n\pi x) dx}{\int_{-1}^1 \cos^2(n\pi x) dx} = \frac{\frac{-x^2}{n\pi} \sin(n\pi x) + \left(\frac{2}{n^3\pi^3}\right) (\sin(n\pi x) - 2n\pi \cos(n\pi x)) + \frac{1}{n\pi} \sin(n\pi x) \Big|_{-1}^1}{\frac{1}{2n\pi} \cos(n\pi x) \sin(n\pi x) + \frac{1}{2} x \Big|_{-1}^1}$$

$$= \frac{-4 \cos(n\pi)}{n^2\pi^2} = \frac{4(-1)^{n+1}}{(n\pi)^2}$$



$$\cos(n\pi) = (-1)^n$$

$$-\cos(n\pi) - (-1)^n = (-1)^{n+1} - (-1)^n = (-1)^{n+1}(-1)^n = (-1)^{2n+1} = -1$$

$$b_n = \frac{\int_{-1}^1 (-x^2 + 1) \sin(n\pi x) dx}{\int_{-1}^1 \sin^2(n\pi x) dx} = \frac{\frac{x^2}{n\pi} \cos(n\pi x) - \left(\frac{1}{n^3\pi^3}\right) (2 \cos(n\pi x) + 2n\pi x \sin(n\pi x)) - \left(\frac{1}{n\pi}\right) \cos(n\pi x) \Big|_{-1}^1}{\frac{-1}{2n\pi} \cos(n\pi x) \sin(n\pi x) + \frac{1}{2} x \Big|_{-1}^1} = \frac{0}{1} = 0$$

$$f(-x) = -f(x)$$

$$-((-x)^2) + 1 = -x^2 + 1$$

$$\sin(\pi/2) = 1 \quad -\sin(\pi/2) = -1$$

$$f(-x) = -\sin(x)$$

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$$c_0 = \frac{2}{3}$$

$$a_n = \frac{4(-1)^{n+1}}{(n\pi)^2}$$

$$b_n = 0$$

$$\frac{4(-1)^{1+1}}{(1\pi)^2} = \frac{4}{\pi^2}$$

$$\frac{4(-1)^{2+1}}{(2\pi)^2} = \frac{-4}{4\pi^2} \quad n=2$$

$$-x^2 + 1 \approx \text{Proj}_S(-x^2 + 1) = c_0(1) + a_1 \cos(\pi x) + b_1 \sin(\pi x) + \dots + a_k \cos(k\pi x) + b_k \sin(k\pi x)$$

So if we project on to the first k of the functions we would have:

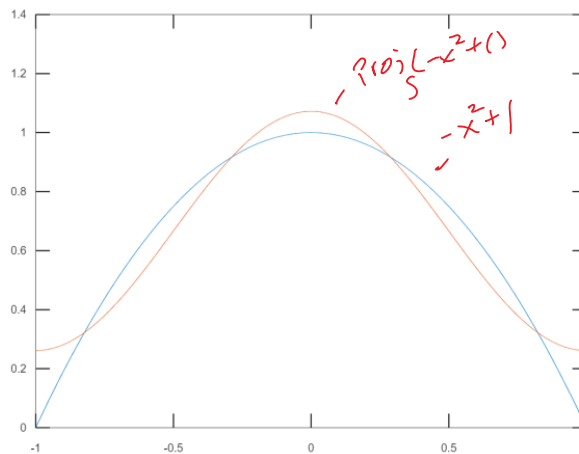
$$\begin{aligned} -x^2 + 1 &\approx \text{Proj}_S(-x^2 + 1) \\ &= \frac{2}{3} + \frac{4}{\pi^2} \cos(\pi x) - \frac{4}{4\pi^2} \cos(2\pi x) + \frac{4}{9\pi^2} \cos(3\pi x) - \frac{4}{16\pi^2} \cos(4\pi x) + \dots + \frac{4(-1)^{k+1}}{k^2\pi^2} \cos(k\pi x) \end{aligned}$$

Let's take $S = \{1, \sin(\pi x), \cos(\pi x)\}$ Then the projection would be

$$\text{Proj}_S(-x^2 + 1) = \frac{2}{3} + \frac{4}{\pi} \cos(\pi x) + O(\sin(\pi x))$$

So the red curve would be the function closest to $-x^2 + 1$ that is in

$$\text{span}\{1, \sin(x), \cos(x)\}$$

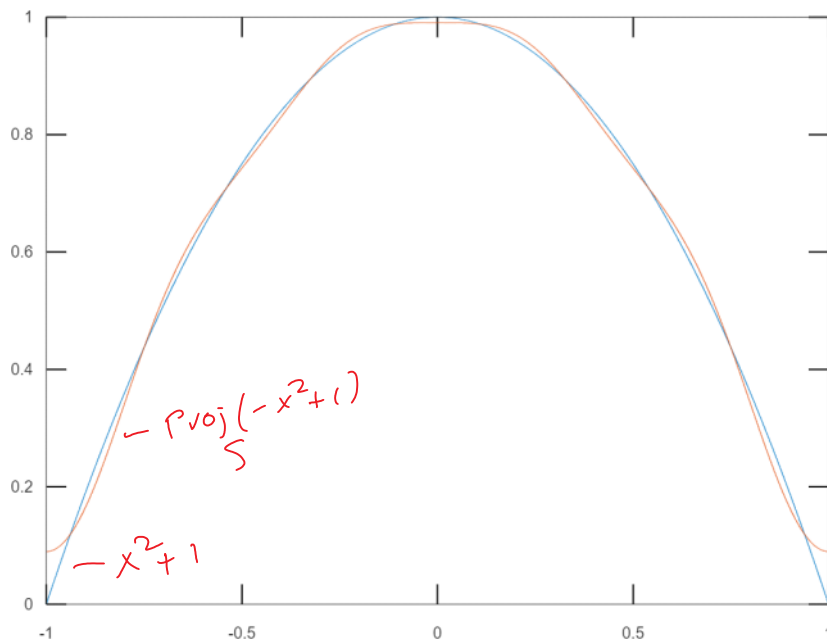


Let's take $\{1, \sin(\pi x), \cos(\pi x), \sin(2\pi x), \cos(2\pi x), \sin(3\pi x), \cos(3\pi x), \sin(4\pi x), \cos(4\pi x)\}$

Then the projection would be:

$$\text{Proj}_S(-x^2 + 1) = \frac{2}{3} + \frac{4}{\pi} \cos(\pi x) - \frac{4}{4\pi} \cos(2\pi x) + \frac{4}{9\pi} \cos(3\pi x) - \frac{4}{16\pi} \cos(4\pi x)$$

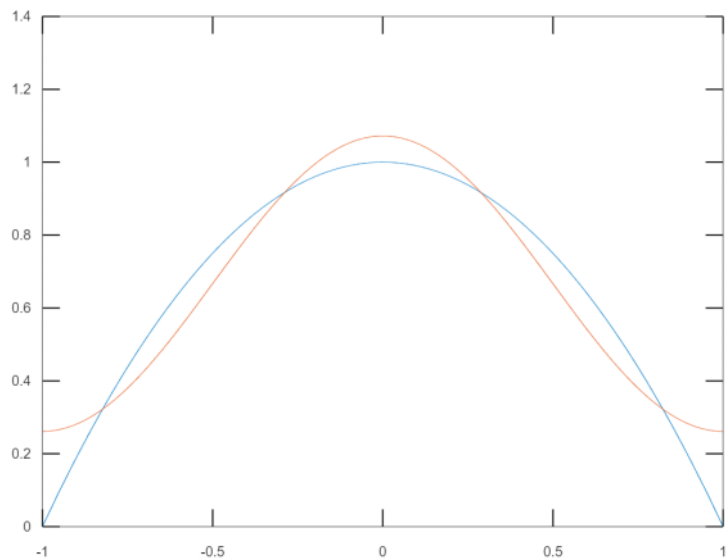
This projection looks much more like the original function of $-x^2 + 1$. What do you think would happen if we take say 100 terms?



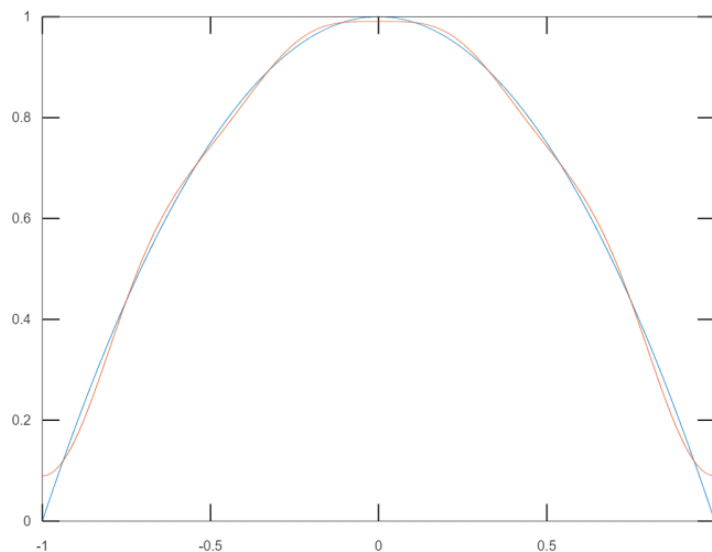
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Various projections of $f(x) = -x^2 + 1$ onto $\text{span}\{1, \cos(\pi x), \sin(\pi x), \cos(2\pi x), \sin(2\pi x), \dots, \cos(k\pi), \sin(k\pi)\}$

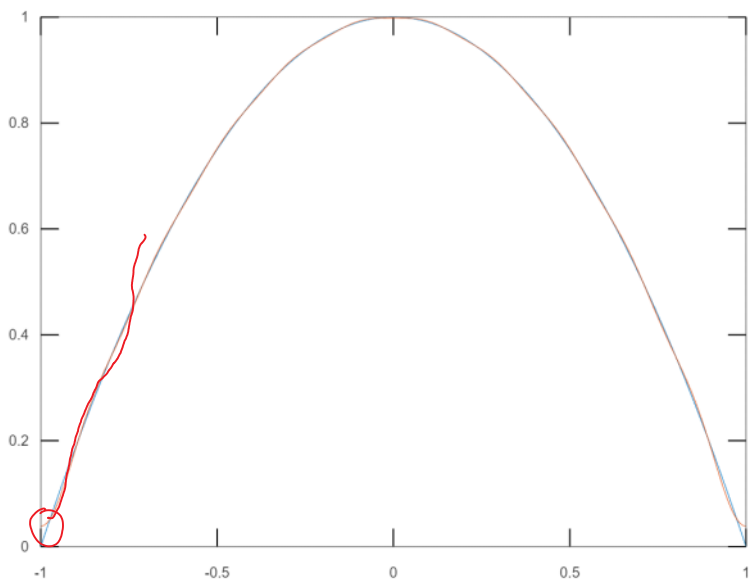
For different values of k



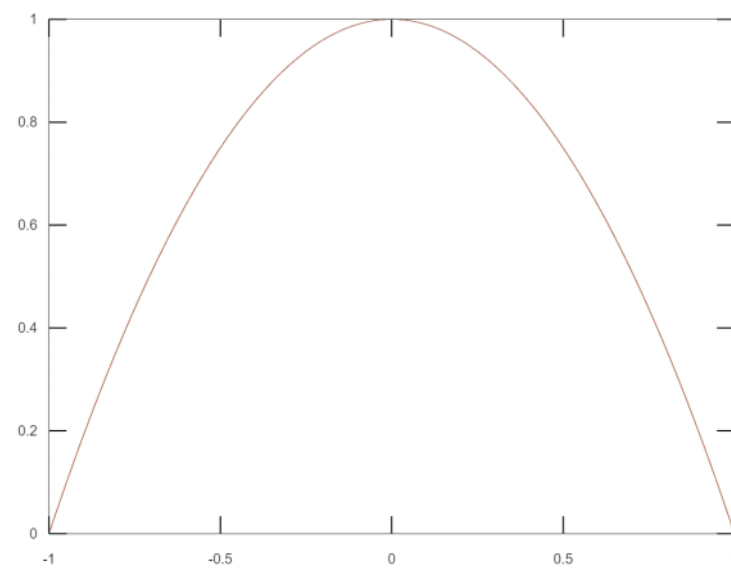
$k = 1$



$k = 4$



$k = 10$



$k = 100$

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Fourier Approximation

The k^{th} Fourier approximate of a function on the interval $[-L, L]$ is the projection of $f(x)$ onto the subspace, S , of $C([-L, L])$ given by:

$$S = \text{span} \left\{ 1, \cos\left(\frac{\pi x}{L}\right), \sin\left(\frac{\pi x}{L}\right), \cos\left(\frac{2\pi x}{L}\right), \sin\left(\frac{2\pi x}{L}\right), \dots, \cos\left(\frac{k\pi x}{L}\right), \sin\left(\frac{k\pi x}{L}\right) \right\}.$$

With inner product $\langle f, g \rangle = \int_{-L}^L f(x) g(x) dx$ the projection has the following form:

$$f(x) \approx \text{Proj}_S(f(x)) = c_0 + \sum_{n=1}^k a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^k b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$c_0 = \frac{\langle 1, f(x) \rangle}{\langle 1, 1 \rangle} = \frac{\int_{-L}^L f(x) dx}{\int_{-L}^L dx} = \frac{1}{2L} \int_{-L}^L f(x) dx$$

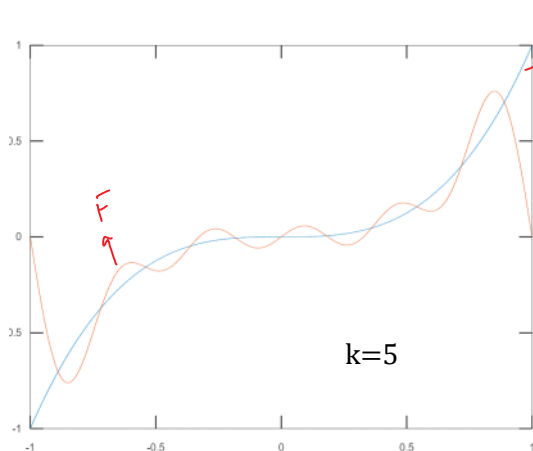
$$a_n = \frac{\langle \cos(n\pi x), f(x) \rangle}{\langle \cos(n\pi x), \cos(n\pi x) \rangle} = \frac{\int_{-L}^L f(x) \cos(n\pi x) dx}{\int_{-L}^L \cos^2(n\pi x) dx} = \frac{1}{L} \int_{-L}^L f(x) \cos(n\pi x) dx$$

$$b_n = \frac{\langle \sin(n\pi x), f(x) \rangle}{\langle \sin(n\pi x), \sin(n\pi x) \rangle} = \frac{\int_{-L}^L f(x) \sin(n\pi x) dx}{\int_{-L}^L \sin^2(n\pi x) dx} = \frac{1}{L} \int_{-L}^L f(x) \sin(n\pi x) dx$$

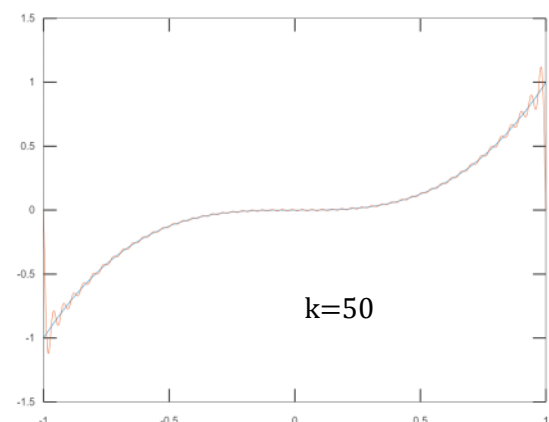
Some Octave code to find Fourier approximations <https://octave-online.net/bucket~9wibCpniiTwpuZGJ8FKBCo>

```
clear %clear variables
L=1; %choose interval
k=5; % choose number of basis functions
X=-L:L/1000:L; % Defining an interval to plot on
c0=(quad (@(x) x.^3, -L, L))*(1/(2*L)); % Find c0 using the quad function to do the integration, this is numerical integration. Think Riemann sums.
F=c0; %set the Fourier approximate to the first term
for n=1:k % Loop through values for n and calculate each coefficient using numerical integration.
    an=(quad (@(x) x.^3.*cos(n*pi*x/L), -L, L))*(1/L); % for cosine terms
    bn=(quad (@(x) x.^3.*sin(n*pi*x/L), -L, L))*(1/L); % for sine terms
    F=F+an.*cos(n*pi*x/L)+bn.*sin(n*pi*x/L); % add to F the new parts of the expansion
end;

plot(X,X.^3) %plot the original function on the interval X (note I capitalized it, @x before made that x a symbol)
hold on % command to overlay the next plot coming on the first
plot(X,F) % plot the fourier approximate on the interval X
hold off % command to stop plot overlay
```



Fourier approximations of $f(x) = x^3$ on $[-1, 1]$



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This actually pops up in lots of applications, one of them is in solving the heat equation. Let's look at the 1-d version.

$u = \text{Temp}$

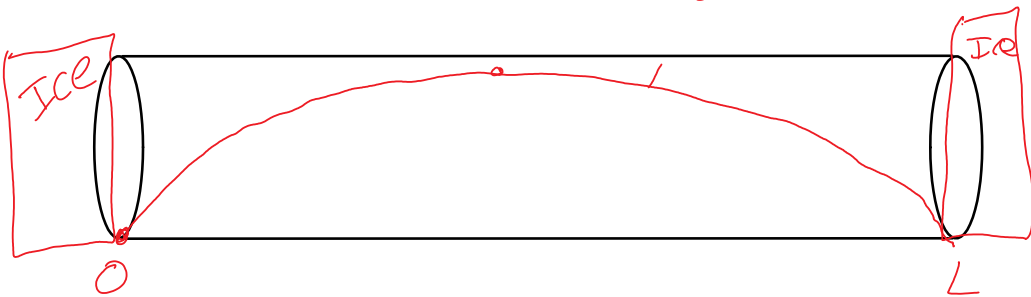
$L=1$

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \text{ for } 0 < x < L \text{ and } t > 0$$

BCs: $u(0, t) = 0$

$u(L, t) = 0$

IC: $u(x, 0) = f(x)$



It turns out that the solution to this system is:

$$u(x, t) = \sum_{n=1}^{\infty} B_n e^{-k \left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L} x\right)$$

While we won't derive it here, those who would like to see this can look at some old lecture notes posted to sakai.

What is not known in the above solution are the coefficients B_n in the series expansion above. However we can find it by looking at the initial condition, the initial temperature profile in the rod, $u(x, 0) = f(x)$.

If we plug in $t = 0$ we get:

$$u(x, 0) = \sum_{n=1}^{\infty} B_n e^{-k \left(\frac{n\pi}{L}\right)^2 (0)} \sin\left(\frac{n\pi}{L} x\right)$$

For an initial temperature profile $f(x)$ this means

$$f(x) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L} x\right)$$

This is the projection of $f(x)$ on to the space S of functions on $[0, 1]$ defined by $S = \text{span}\left\{\sin\left(\frac{x\pi}{L}\right), \sin\left(\frac{2\pi x}{L}\right), \sin\left(\frac{3\pi x}{L}\right), \dots\right\}$

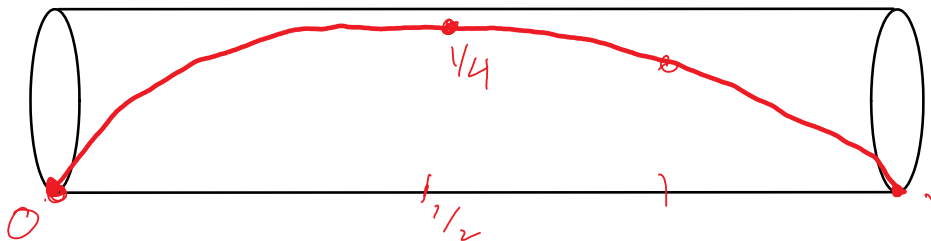
This time with inner product $\langle f, g \rangle = \int_0^1 f g dx$ with respect to which the sin basis functions above are orthogonal.

If we could add infinitely many terms we could get an exact representation of f , but the best we can do here is take a finite approximation, however in some cases f may be in the span of a finite set of the sin terms above, in that case we can indeed get an exact answer. Either way we have a formula for those coefficients!

$$B_n = \frac{\langle \sin(n\pi x), f(x) \rangle}{\langle \sin(n\pi x), \sin(n\pi x) \rangle} = \frac{\int_0^1 f(x) \sin(n\pi x) dx}{\int_0^1 \sin^2(n\pi x) dx} = 2 \int_0^1 f(x) \sin(n\pi x) dx$$

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Suppose our interval is $[0,1]$ and initial temperature profile is $f(x) = x(1-x) \sim -x^2 + x$



Then we would compute the coefficients using our formula

$$B_n = 2 \int_0^1 (-x^2 + x) \sin(n\pi x) dx = \frac{2}{n^3\pi^3} - \frac{\cos(n\pi)}{n^3\pi^3} = \frac{2 - (-1)^n}{n^3\pi^3}$$

And then our solution

$$u(x, t) = \sum_{n=1}^{\infty} B_n e^{-k\left(\frac{n\pi}{1}\right)^2 t} \sin\left(\frac{n\pi}{1} x\right)$$

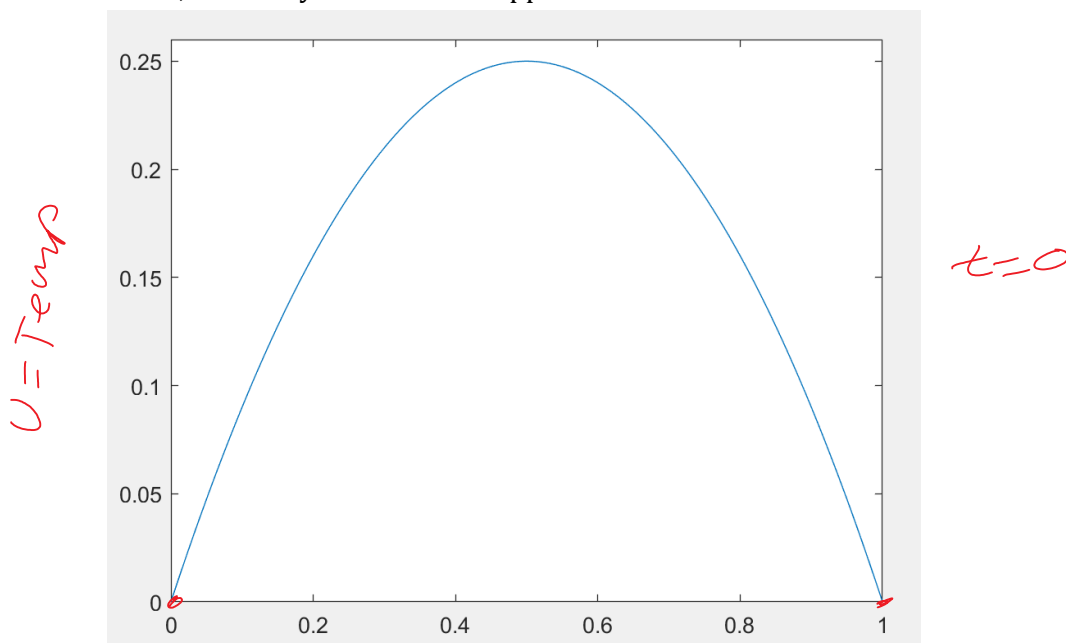
becomes

$$u(x, t) \approx \sum_{n=1}^{100} \frac{2 - (-1)^n}{n^3\pi^3} e^{-k(n\pi)^2 t} \sin(n\pi x)$$

We can approximate with say 100 terms

Let's watch it evolve in time!

Here it is at $t = 0$, what do you think will happen as t increases?



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What if the function that we are looking for happens to be in the span of a finite number of the basis functions?

Let S be a non zero subspace of an inner product space with orthogonal basis $\{\vec{v}_1 \dots \vec{v}_k\}$, and let $\vec{u} \in \mathbb{R}^n$. Then,

(a) $\text{proj}_S \vec{u}$ is in S .

(b) $\vec{u} - \text{proj}_S \vec{u}$ is orthogonal to S .

(c) if $\vec{u}$ is in S then $\text{proj}_S \vec{u} = \vec{u}$.

(d) $\text{proj}_S \vec{u}$ is independent of the choice of orthogonal basis for S .

Suppose this time for our solution to the 1-d heat equation on $[0, L]$

$$u(x, t) = \sum_{n=1}^{\infty} B_n e^{-k \left(\frac{n\pi}{L} \right)^2 t} \sin \left(\frac{n\pi}{L} x \right)$$

The initial temperature profile was given by:

$$u(x, 0) = 4 \sin \left(\frac{\pi x}{L} \right) + 6 \sin \left(\frac{3\pi x}{L} \right) + 3 \sin \left(\frac{100\pi x}{L} \right)$$

What would the solution be then? $n=1$ $n=3$ $n=100$

$$u(x, 0) = \sum_{n=1}^{\infty} B_n \sin(n\pi x) = 4 \sin(\pi x) + 6 \sin(3\pi x) + 3 \sin(100\pi x)$$

$n=1$ $n=3$ $n=100$

$$\begin{aligned} & B_1 \sin(\pi x) + B_3 \sin(3\pi x) + B_{100} \sin(100\pi x) \\ &= 4 \sin(\pi x) + 6 \sin(3\pi x) + 3 \sin(100\pi x) \end{aligned}$$

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Another use of inner products is that of the weighted least squares regression. As a hypothetical, suppose you are measuring conductivity σ of a soil sample as a function of porosity ϕ .

If you recall Archies law suggests $\sigma = a\phi^m$ which we can make a linear equation as

$$\ln(\sigma) = m \ln(\phi) + \ln(a)$$

Further suppose that you know that your instrument is ten times as accurate when the conductivity measured is in the range $[10, 50]$ naturally in any least squares regression problem to find m and a we would want to trust measurements in that range more than those outside that range.

We know that the least squares solution can be found by solving the problem

$$A^T A \vec{x} = A^T \vec{y} \quad \longleftrightarrow \quad A \vec{x} = \text{proj}_{\text{col}(A)}(\vec{y})$$



We also know that projections onto subspaces come from inner products, we could use the dot product as before, but we don't have to.

Let S be a subspace of an inner product space $(V, \langle \cdot, \cdot \rangle)$ and suppose that S has an orthogonal basis $\{\vec{v}_1 \dots \vec{v}_k\}$. Then the projection of $\vec{v}$ onto S is given by

$$\text{Proj}_S(\vec{v}) = \frac{\langle \vec{v}_1, \vec{v} \rangle}{\langle \vec{v}_1, \vec{v}_1 \rangle} \vec{v}_1 + \frac{\langle \vec{v}_2, \vec{v} \rangle}{\langle \vec{v}_2, \vec{v}_2 \rangle} \vec{v}_2 + \dots + \frac{\langle \vec{v}_k, \vec{v} \rangle}{\langle \vec{v}_k, \vec{v}_k \rangle} \vec{v}_k$$

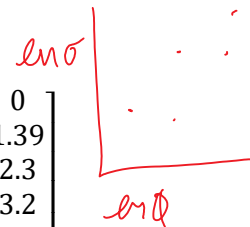
Instead we could use a weighted inner product that weights points in the range $[10, 50]$ ten times as much as those not in that range.

Suppose our data is the following

$$\phi = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \\ 0.5 \\ 0.8 \end{bmatrix} \quad \sigma = \begin{bmatrix} 8 \\ 4 \\ 10 \\ 25 \\ 60 \end{bmatrix}$$

$$\ln(\phi) = \begin{bmatrix} -2.3 \\ -1.6 \\ -1.2 \\ -0.69 \\ -0.22 \end{bmatrix}$$

$$\ln(\sigma) = \begin{bmatrix} 0 \\ 1.39 \\ 2.3 \\ 3.2 \\ 4.09 \end{bmatrix}$$



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Our system of equations with no solution is

$$m \ln(\phi) + \ln(a) = \ln(\sigma)$$

$$\begin{bmatrix} -2.3 & 1 \\ -1.6 & 1 \\ -1.2 & 1 \\ -0.69 & 1 \\ -0.22 & 1 \end{bmatrix} \begin{bmatrix} m \\ \ln(a) \end{bmatrix} = \begin{bmatrix} 2.08 \\ 0.69 \\ 2.3 \\ 3.2 \\ 4.09 \end{bmatrix}$$

$\downarrow \quad \downarrow$
 $a_1 \quad a_2$

$$\ln(\phi) = \begin{bmatrix} -2.3 \\ -1.6 \\ -1.2 \\ -0.69 \\ -0.22 \end{bmatrix} \quad \ln(\sigma) = \begin{bmatrix} 0 \\ 1.39 \\ 2.3 \\ 3.2 \\ 4.09 \end{bmatrix}$$

Steps: solving using the regular dot product as our inner product

- 1. get an orthonormal basis for $\text{col}(A)$ using Gram – Schmidt
2. Project y onto that basis
- 3 Solve $Ax = \text{proj}_{\text{col}(A)}(y)$

Octave code

```
%define data
p=[0.1;0.2;0.3;0.5;0.8]
s=[8;2;10;25;1000]
x=log(p)
y=log(s)
%define A
A1=[x, ones(5,1)]
%gram-Schmidt
A(:,1)=A1(:,1);
A(:,2)=A1(:,2)-(dot(A(:,1),A1(:,2))/dot(A(:,1),A(:,1)))*A(:,1);
%normalize them with respect to our inner product, simplifies the projection formula
A(:,1)=(1/sqrt(dot(A(:,1),A(:,1))))*A(:,1);
A(:,2)=(1/sqrt(dot(A(:,2),A(:,2))))*A(:,2);
%project y
ydot=dot(A(:,1),y)*A(:,1)+dot(A(:,2),y)*A(:,2)
%set up Ax=proj_col(a)(y) system
Als=[A1 ydot]
%solve the system
xls=rref(Als)
%plot data and solution
X=[-3:0.001:0];
plot(x,y,'o')
hold on
plot(X,xls(1,3)*X+xls(2,3))
hold off
% note with reg dot product we get what we would from the normal equations
disp('what we would get from normal equations, the same!')
inv(A1'*A1)*A1'*y
clear A
```

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Now let's choose a weighted inner product, here we have 5 data points and the 3 and fourth are in the more trusted range....

$$\phi = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.3 \\ 0.5 \\ 0.8 \end{bmatrix} \quad \sigma = \begin{bmatrix} 1 \\ 4 \\ 10 \\ 25 \\ 60 \end{bmatrix} \quad \ln(\phi) = \begin{bmatrix} -2.3 \\ -1.6 \\ -1.2 \\ -0.69 \\ -0.22 \end{bmatrix} \quad \ln(\sigma) = \begin{bmatrix} 0 \\ 1.39 \\ 2.3 \\ 3.2 \\ 4.09 \end{bmatrix}$$

So we might instead take the inner product to be:

$$\langle \vec{x}, \vec{y} \rangle = x_1 y_1 + x_2 y_2 + 10x_3 y_3 + 10x_4 y_4 + x_5 y_5$$

We could write an octave function that does this in the following way

```
function c=wdot(x,y)
c=x(1)*y(1)+x(2)*y(2)+10*(x(3)*y(3))+10*(x(4)*y(4))+x(5)*y(5);
```

Then we can follow the exact same procedure as before but replacing dot with wdot!

Steps: solving using the regular dot product as our inner product

1. get an orthonormal basis for $\text{col}(A)$ using Gram – Schmidt
2. Project y onto that basis
- 3 Solve $Ax = \text{proj}_{\text{col}(A)}(y)$

$$\langle \vec{x}, \vec{y} \rangle \rightarrow \langle \vec{y}, \vec{y} \rangle$$

% now let's do the same using our weighted inner product.
% get an orthogonal basis with respect to our inner product using gram schmidt.

```
A(:,1)=A1(:,1);
A(:,2)=A1(:,2)-(wdot(A(:,1),A1(:,2))/wdot(A(:,1),A(:,1)))*A(:,1);
%normalize them with respect to our inner product to simplify projection formula
```

```
A(:,1)=(1/sqrt(wdot(A(:,1),A(:,1))))*A(:,1);
A(:,2)=(1/sqrt(wdot(A(:,2),A(:,2))))*A(:,2);
```

```
%project y with our weighted inner product
ywdot=wdot(A(:,1),y)*A(:,1)+wdot(A(:,2),y)*A(:,2)
```

```
% Set up Ax=proj_col(a)(y)
```

```
Awls=[A1 ywdot]
```

```
%solve the system
```

```
xwls=rref(Awls)
```

```
%plot the solution with the data
```

```
figure
```

```
plot(x,y,'o')
```

```
hold on
```

```
plot(X,xwls(1,3)*X+xwls(2,3))
```

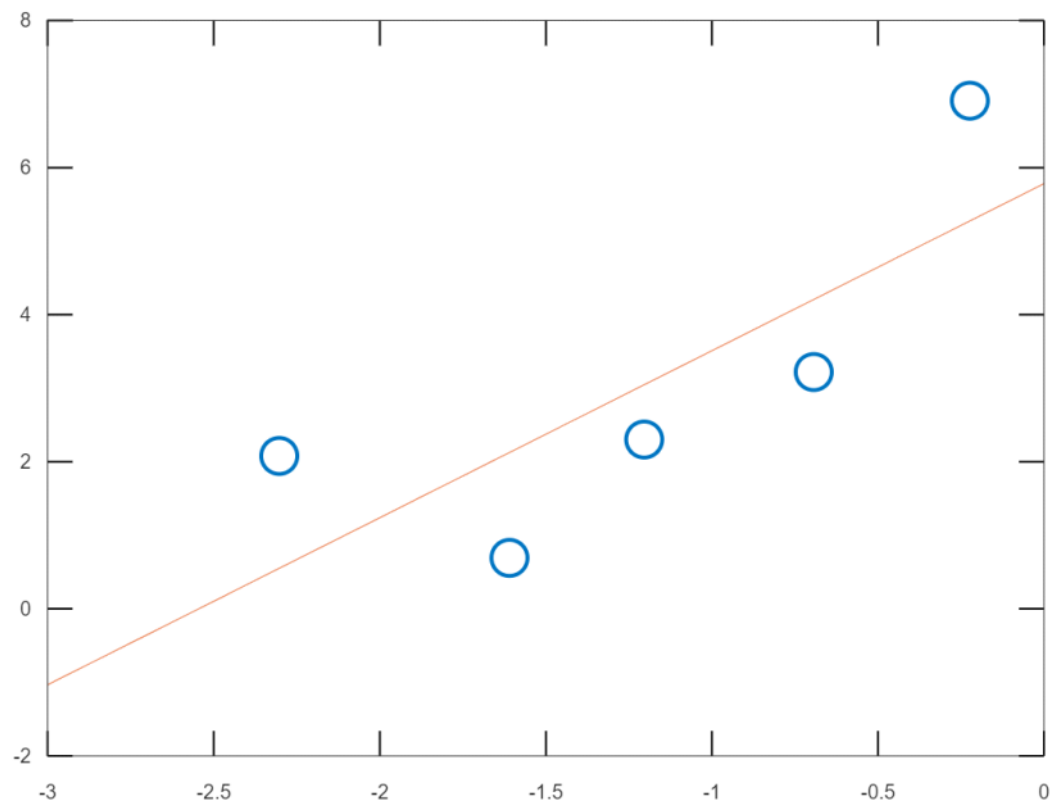
```
hold off
```

$$\text{wdot}(A(:,1), A(:,1)) = /$$

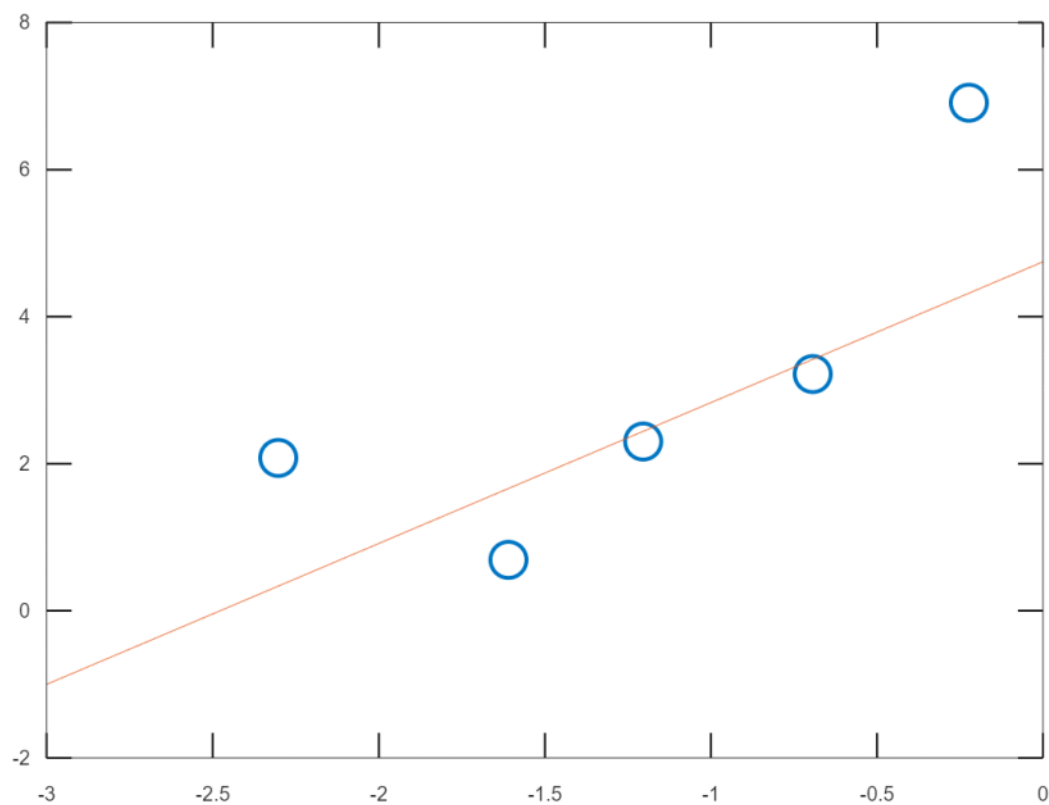
Let's see how the solutions are different head to <https://octave-online.net/bucket~DbiUkRmpkH4VAg5Grocle2>
If you want to follow along

Section 10.3 Applications of Inner Products

Regular dot product solution, all points weighted equally



Solution where we weight the 3rd and 4th points 10 times as much



Quiz 12 Due Friday 4/26 11:00pm

For your quiz this Thursday we won't do any in class work. I would rather spend the time reviewing for the final. However, you will still need to submit something to Gradescope for credit. Here is what you need to submit.

On Sakai you will find a Quiz12.docx and a Quiz12.pdf, you can choose which you wish to use, but I think the Quiz12.docx will be easier to deal with, then you can export your quiz as a pdf and upload it to Gradescope.

There will just be two questions there, the first will ask you about what your major is and what your plans for the future are. I am interested to know what kinds of applications for mathematics you might have coming up in the future. This would help me find better ways to connect mathematical ideas to real world applications when teaching. This is totally optional! You can just leave it blank if you don't want to share that information, no harm no foul. The second question will just ask for two plots of a Fourier series approximation of a function of your choice. I would recommend just using the code here <https://octave-online.net/bucket~9wibCpniiTwpuZGj8FKBCo> and inputting your own function. I mostly just want you to play around with them and be mystified at the power of mathematics.

On Thursday we will just review for the final. I will post a few practice problems relating to the material covered since exam 2 shortly. What you want to study will be those problems **in addition to** the practice problems for exam 1 and exam 2.

Otherwise, just as a reminder the final exam is in our class room on May 3rd at 8:00am

Happy Almost Summer!