

Section 2.1 Vectors in R^n

In general vectors are ordered list of mathematical objects. One could have a list of real numbers, complex numbers, functions, matrices.... Some of which we will touch on later in this course. For now we will introduce the concept of vectors using real numbers.

A vector $\vec{v} \in R^n$ is an ordered list of n real numbers and are written in many different ways...

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix} \quad \vec{v} = \langle v_1, v_2, v_3, \dots, v_n \rangle \quad \vec{v} = (v_1, v_2, v_3, \dots, v_n) \quad \vec{v} = v_1 e_1 + v_2 e_2 + v_3 e_3 + \dots + v_n e_n$$

Vectors themselves are mathematical objects and as mathematicians that makes us want to know when they are equal, how to add them, how to multiply them and what we can do with them.

When two vectors are equal:

Are the vectors $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ equal? What about $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$?

equal!

Not equal

Two vectors $\vec{u}$ and $\vec{v} \in R^n$ are equal when $u_1 = v_1, u_2 = v_2, u_3 = v_3, \dots, u_n = v_n$

Vector Addition and Subtraction: Let $\vec{u}, \vec{v}$ be vectors in R^n then:

$$\vec{u} + \vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ u_3 + v_3 \\ \vdots \\ u_n + v_n \end{bmatrix} \quad \vec{u} - \vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix} - \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 - v_1 \\ u_2 - v_2 \\ u_3 - v_3 \\ \vdots \\ u_n - v_n \end{bmatrix}$$

Ex: Calculate $\vec{u} + \vec{v}$ if $\vec{u} = \langle 1, -2, 3, 0 \rangle$ and $\vec{v} = \langle 4, 1, -2, 1 \rangle$

$$\langle 1, -2, 3, 0 \rangle + \langle 4, 1, -2, 1 \rangle = \langle 1+4, -2+1, 3+(-2), 0+1 \rangle = \langle 5, -1, 1, 1 \rangle$$

Ex: Calculate $\vec{u} - \vec{v}$ if $\vec{u} = \langle 1, -2, 3, 0 \rangle$ and $\vec{v} = \langle 4, 1, -2, 1 \rangle$

$$\langle 1-4, -2-1, 3-(-2), 0-1 \rangle = \langle -3, -3, 5, -1 \rangle$$

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Vector Multiplication is a bit more subtle and can be done in different ways. Which ways are useful depends on the problem at hand and whether or not the operation has desirable mathematical properties.

The Hadamard Product: This is a natural vector product one might immediately think of

$$\vec{u} \circ \vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix} \circ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 v_1 \\ u_2 v_2 \\ u_3 v_3 \\ \vdots \\ u_n v_n \end{bmatrix}$$

This particular product is useful in some computational applications however as we will see in a moment in this class we will be most concerned with vector operations that say something about geometry. It is good to mention this one though.

Scalar Multiplication let $a \in R$ and $\vec{u} \in R^n$ then:

$$a\vec{u} = a \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} au_1 \\ au_2 \\ au_3 \\ \vdots \\ au_n \end{bmatrix}$$

This product is of a single real number (a scalar) and the vector where we simply multiply each component by the scalar. Geometrically this product changes the length of the vector ("scales it").

Example: calculate $3\langle -1, 2, 4 \rangle$

$$= \langle 3(-1), 3(2), 3(4) \rangle = \langle -3, 6, 12 \rangle$$

The dot product: let $\vec{u} \in R^n$ $\vec{v} \in R^n$ then

$$\vec{u} \cdot \vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + u_3 v_3 + \cdots + u_n v_n$$

This product takes two vectors in R^n to a single number in R . We will focus in on this very important product in chapter 8 when we talk about orthogonality. As it turns out when the dot product of two vectors is zero they are at 90° to each other. It can also be used to determine how much of one vector is in the direction of the other.

Example: let $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -1 \\ 1 \\ 3 \\ 2 \end{bmatrix}$ calculate $\vec{u} \cdot \vec{v}$

$$\begin{aligned} \vec{u} \cdot \vec{v} &= (1)(-1) + 2(1) + (-1)(3) + 0(2) \\ &= -1 + 2 - 3 + 0 = \boxed{-2} \end{aligned}$$

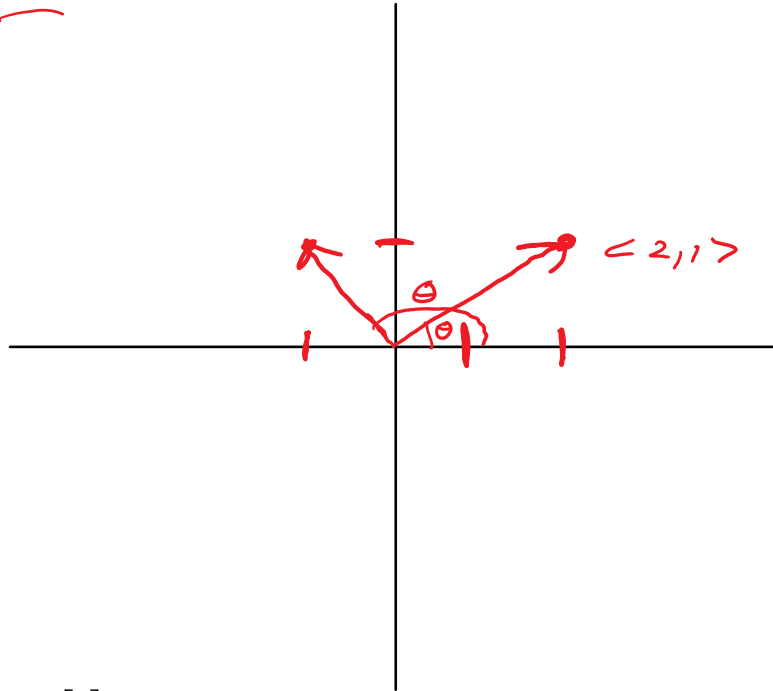
This list is not exhaustive...

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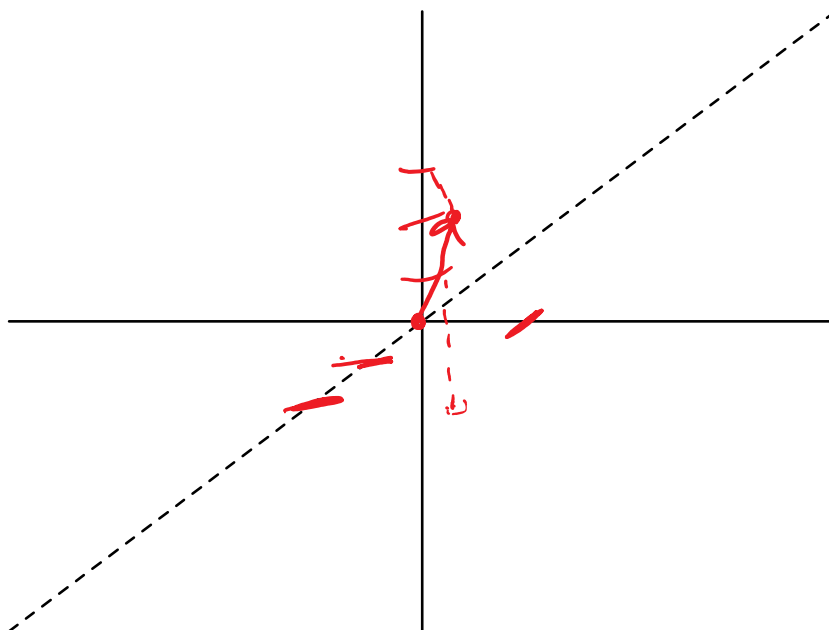
Geometry of Vectors

This may or may not be your first introduction to vectors, especially if you have had a physics class. This because we think of vectors as objects which have a magnitude and a direction. For example representing the velocity of a plane, the magnitude is the speed and direction where it is headed. We often represent this with an arrow pointing in a specific direction whose length is equal to the magnitude of the vector.

Ex: Draw the vectors $\vec{u} = \langle 2, 1 \rangle$ and $\vec{v} = \langle -1, 1 \rangle$



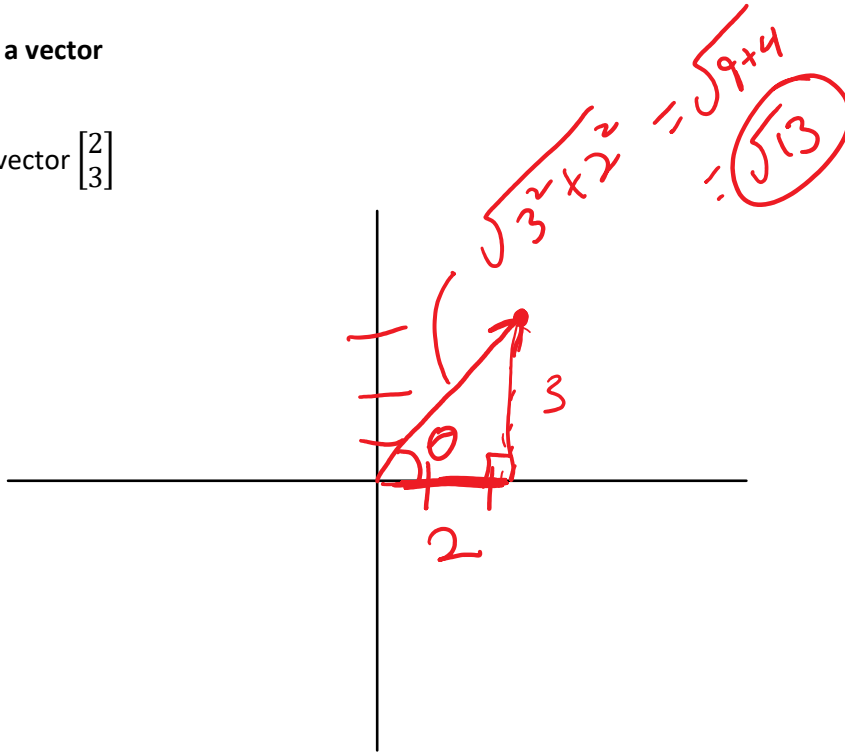
Example: Draw the vector $\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$



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Magnitude (length) of a vector

Find the length of the vector $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$



In general you can find the length of a vector by taking the square root of the sum of the squares of the components.

Magnitude of a vector let $\vec{u} \in \mathbf{R}^n$ then the magnitude is $|\vec{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2 \cdots u_n^2}$

Example: find the magnitude of $\vec{u} = \langle 1, -2, 0, 1, 4, -1 \rangle$

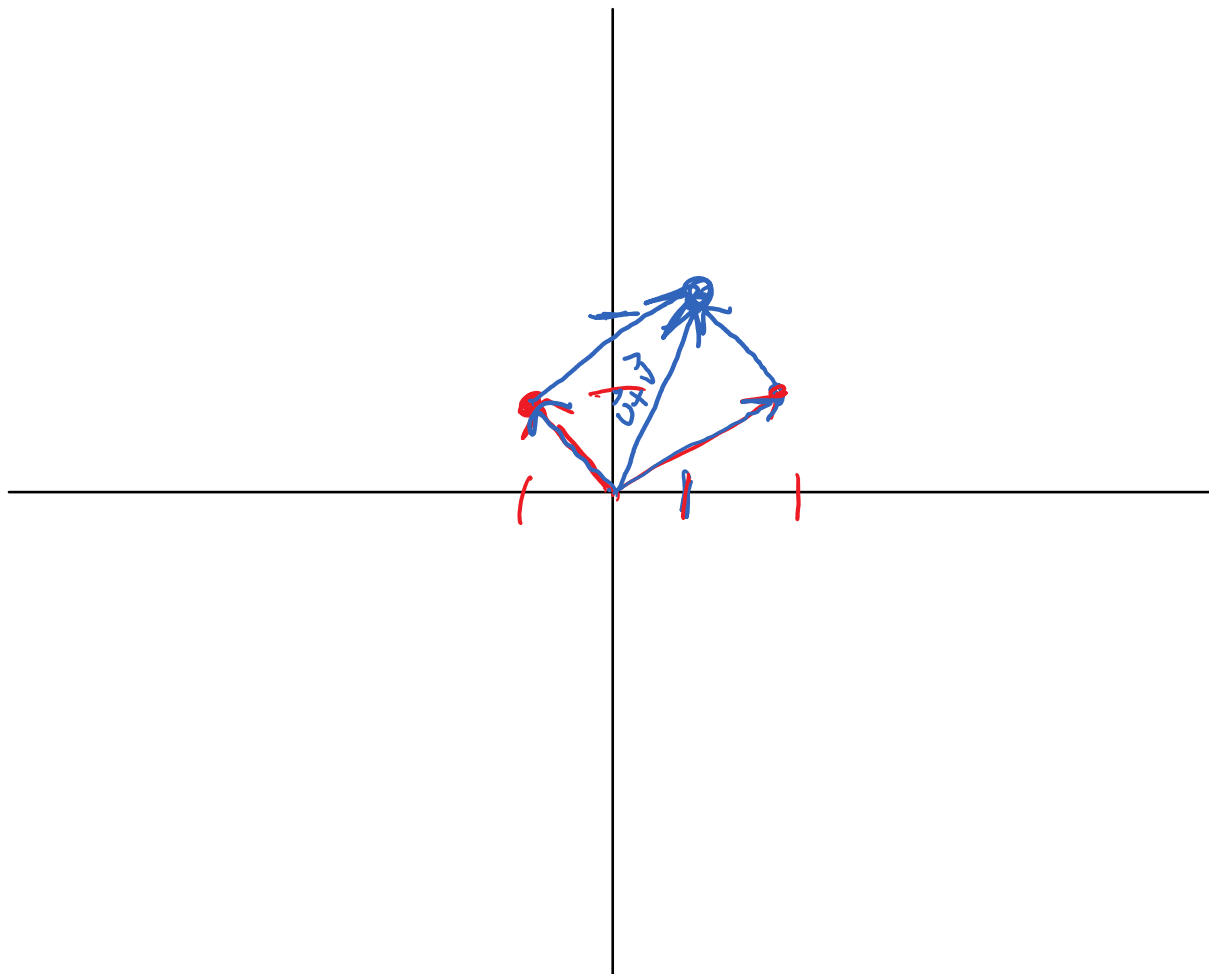
$$\begin{aligned} \|\vec{u}\| &= \sqrt{(1)^2 + (-2)^2 + (0)^2 + (1)^2 + (4)^2 + (-1)^2} \\ &= \sqrt{1 + 4 + 0 + 1 + 16 + 1} \\ &= \sqrt{23} \end{aligned}$$

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Geometry of Vector Addition

Vector addition can be thought of as following one vector from tail to tip and then attaching the second vector's tail to that tip and moving along the second vector.

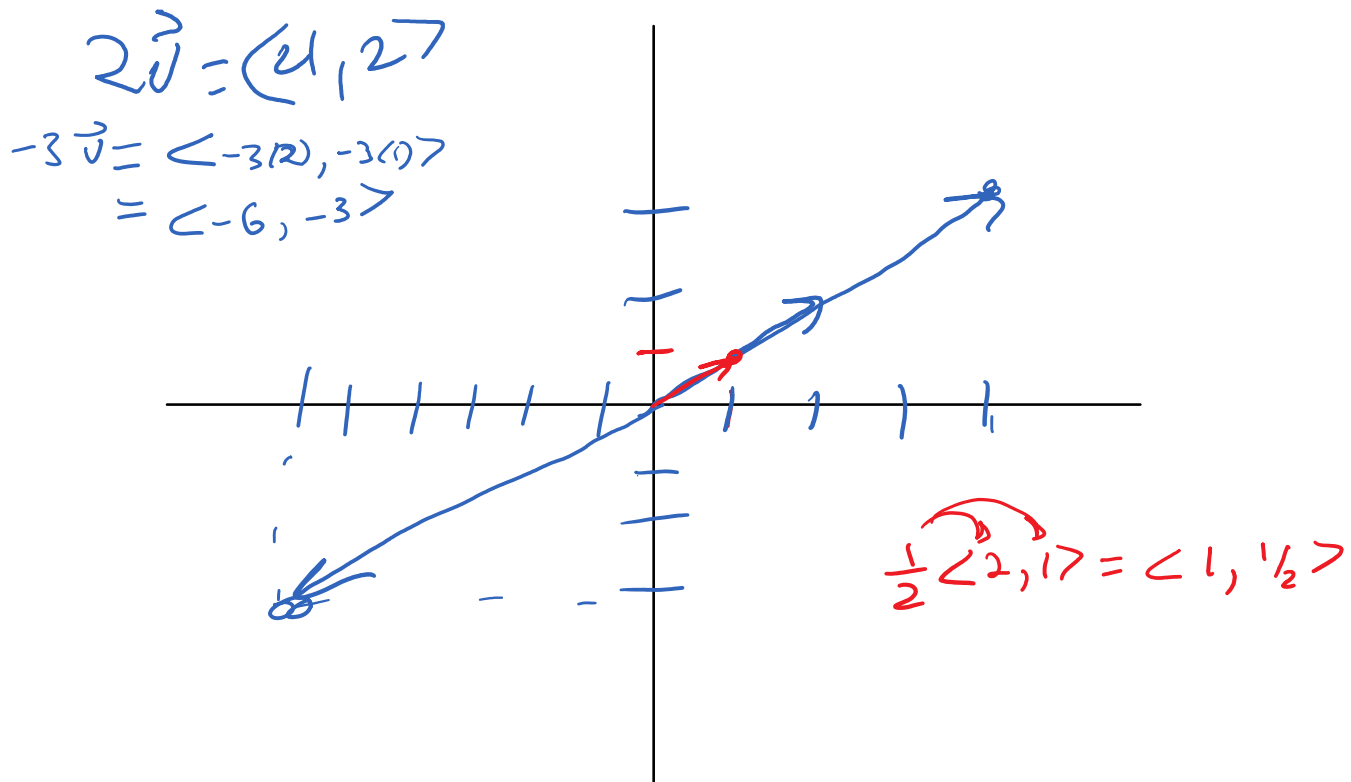
Example: Draw $\vec{u} = \langle 2, 1 \rangle$, $\vec{v} = \langle -1, 1 \rangle$ and $\vec{u} + \vec{v} = \langle 1, 2 \rangle$



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Geometry of Scalar Multiplication Scalar multiplication does not change the direction of a vector only it's magnitude!

Example: Draw $\vec{u} = \langle 2, 1 \rangle$, $2\vec{u}$, $-3\vec{u}$, $\frac{1}{2}\vec{u}$ and calculate the magnitude of each vector.



$$|\vec{u}| = \sqrt{2^2 + 1^2} = \sqrt{5}$$

$$\begin{aligned} |2\vec{u}| &= \sqrt{4^2 + 2^2} = \sqrt{16 + 4} = \sqrt{20} \\ &= \sqrt{4(5)} \\ &= \sqrt{4}\sqrt{5} \\ &= 2\sqrt{5} \end{aligned}$$

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Linear Combinations We will use this phrase a lot in this course, a linear combination of vectors is a sum of scalar multiples of vectors.

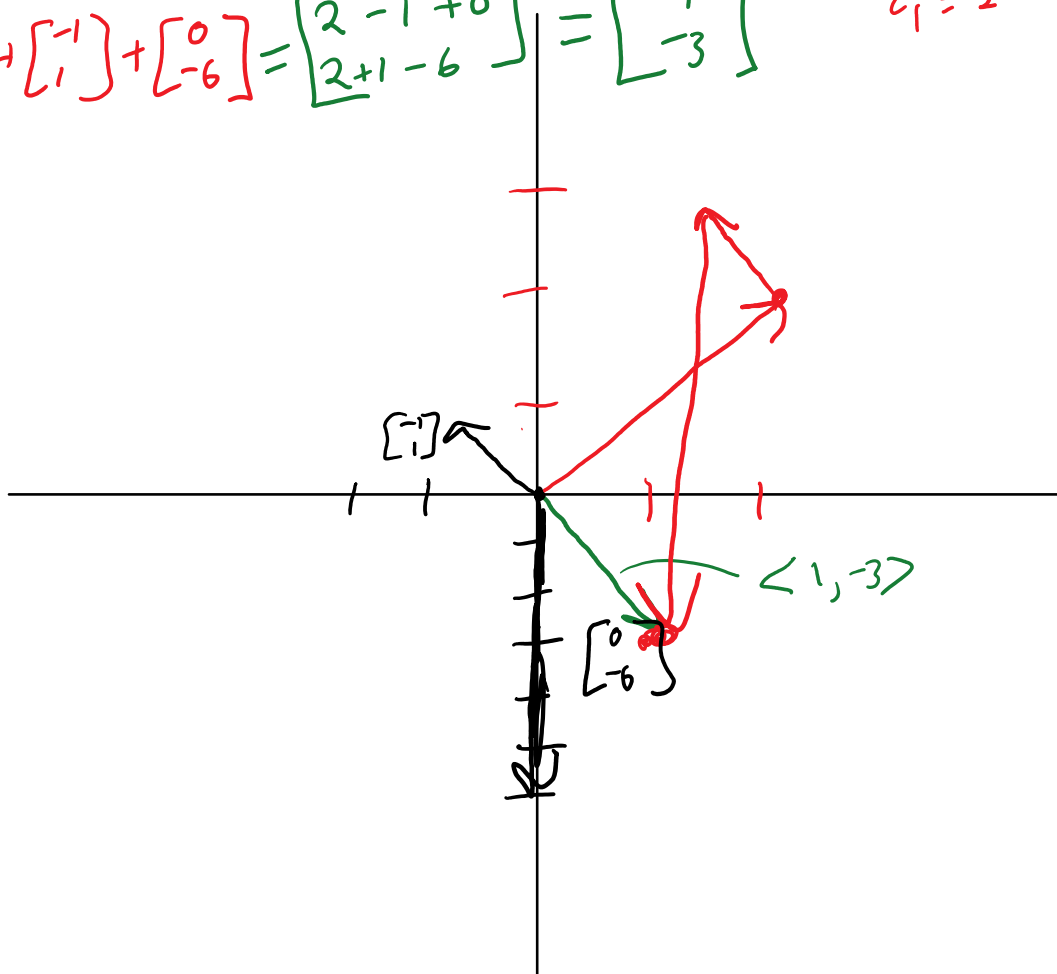
If $\vec{u}_1, \vec{u}_2, \vec{u}_3 \dots \vec{u}_m$ are vectors in R^n and $c_1, c_2, c_3 \dots c_m$ scalars in R then a linear combination of those vectors is a sum of the form:

$$\underline{c_1} \underline{\vec{u}_1} + \underline{c_2} \underline{\vec{u}_2} + c_3 \vec{u}_3 + \dots c_m \vec{u}_m$$

Example: Draw the linear combination $2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} + -3 \begin{bmatrix} 0 \\ 2 \end{bmatrix}$

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \vec{u}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \vec{u}_3 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$
$$c_1 = 2 \quad c_2 = 1 \quad c_3 = -3$$

$$\rightarrow \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ -6 \end{bmatrix} = \begin{bmatrix} 2-1+0 \\ 2+1-6 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$$



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Representing systems of equations as linear combinations

We can use our new concepts to represent linear systems of equations which can also provide a different lens on what we mean by a solution to a linear system

Consider the systems shown below and show one represents the other

$$\begin{cases} x_1 + x_2 - 2x_3 = 3 \\ -5x_1 + 7x_2 + 6x_3 = 12 \\ 4x_1 + 0x_2 - 8x_3 = 0 \end{cases}$$

$$x_1 \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 7 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 6 \\ -8 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ -5x_1 \\ 4x_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ 7x_2 \\ 0x_2 \end{bmatrix} + \begin{bmatrix} -2x_3 \\ 6x_3 \\ -8x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} x_1 + x_2 - 2x_3 \\ -5x_1 + 7x_2 + 6x_3 \\ 4x_1 + 0x_2 - 8x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ 0 \end{bmatrix}$$

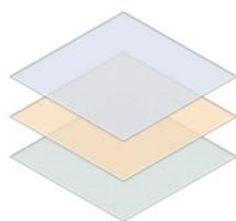
$$x_1 \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 7 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 6 \\ -8 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ 0 \end{bmatrix}$$

Geometric interpretations

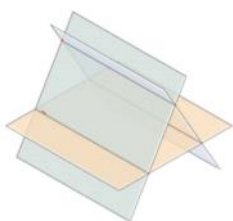
$$\begin{cases} x_1 + x_2 - 2x_3 = 3 \\ -5x_1 + 7x_2 + 6x_3 = 12 \\ 4x_1 + 0x_2 - 8x_3 = 0 \end{cases}$$

Intersection of planes

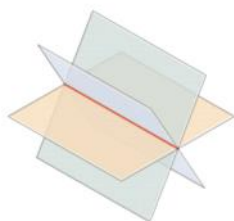
Scalars needed so that a linear combination reaches a given vector



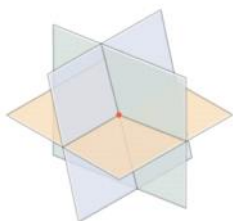
(a) Parallel planes, no points in common to all three.



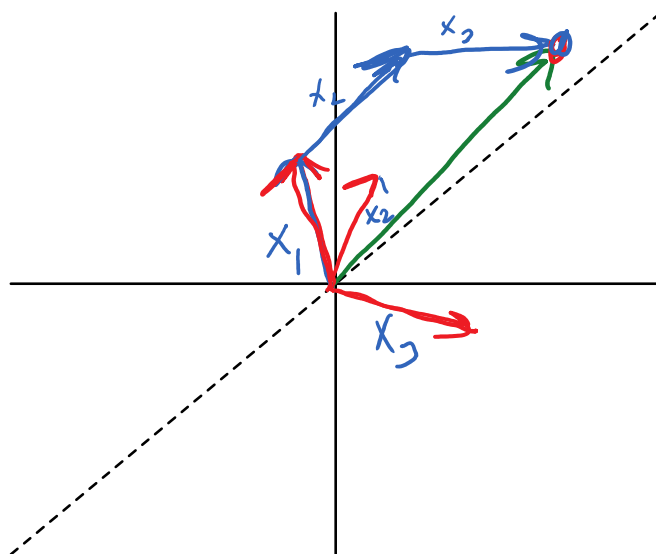
(b) Planes intersect in pairs, no points in common to all three.



(c) Planes intersect in a line, infinitely many points in common.



(d) Planes intersect at a point, unique point in common.



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The solution to this system

$$x_1 \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 7 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 6 \\ -8 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ 0 \end{bmatrix}$$

turns out to be $x_1 = \frac{9}{2}, x_2 = 3, x_3 = \frac{9}{4}$

$$\text{That is } \frac{9}{2} \begin{bmatrix} 1 \\ -5 \\ 4 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 7 \\ 0 \end{bmatrix} + \frac{9}{4} \begin{bmatrix} -2 \\ 6 \\ -8 \end{bmatrix} = \begin{bmatrix} 3 \\ 12 \\ 0 \end{bmatrix}$$

What about a system where we have infinitely many solutions?

$$x + y + z = 2$$

$$-2x - 2y + 2z = 4$$

$$3z = 6$$

$$x_1 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

A =

$$\begin{bmatrix} 1 & 1 & 1 & 2 \\ -2 & -2 & 2 & 4 \\ 0 & 0 & 3 & 6 \end{bmatrix}$$

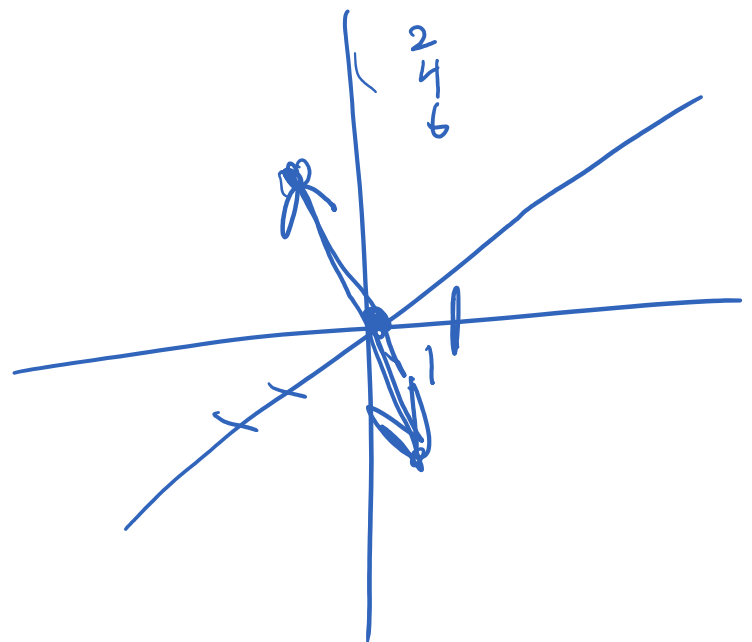
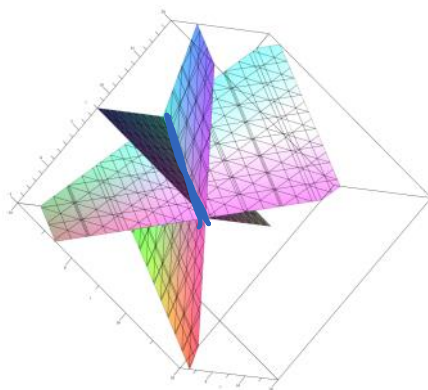
>> rref(A)

ans =

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 &= -t \\ x_2 &= t \\ x_3 &= 2 \end{aligned}$$

$$-t \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$



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Fertilizer is sold in bags labelled with three numbers that indicate the amount of nitrogen (N), phosphoric acid (P_2O_5), and potash (K_2O) present. A bag of *Vigoro Ultra Turf* has the numbers "29-3-4," which means that 100 pounds of this fertilizer contains 29 pounds of nitrogen, 3 pounds of phosphoric acid, and 4 pounds of potash. A bag of *Parker's Premium Starter* has 18 pounds of nitrogen, 25 pounds of phosphoric acid, and 6 pounds of potash per 100 pounds. (See the discussion at the start of the section.)

Determine the amount of each type required to yield a mixture containing the given amounts of nitrogen, phosphoric acid, and potash.

224 pounds of nitrogen, 162 pounds of phosphoric acid, and 52 pounds of potash.

We can think of the total amount of each molecule as a linear combination of the amounts we get from each bag

1 bag of ultra turf has 29 lbs of N, 3 lbs P_2O_5 , 4 lbs K_2O

1 bag of Premium Starter has 18 lbs of N, 25 lbs P_2O_5 , 6 lbs K_2O

so as vectors

$$\text{ultra-turf (UT)} \begin{bmatrix} 29 \\ 3 \\ 4 \end{bmatrix} \quad \text{Premium Starter (PS)} \begin{bmatrix} 18 \\ 25 \\ 6 \end{bmatrix}$$

we want x_1 bags of UT and x_2 bags of PS so that

$$x_1 \begin{bmatrix} 29 \\ 3 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 18 \\ 25 \\ 6 \end{bmatrix} = \begin{bmatrix} 224 \\ 162 \\ 52 \end{bmatrix}$$

or

$$\begin{aligned} 29x_1 + 18x_2 &= 224 \\ 3x_1 + 25x_2 &= 162 \\ 4x_1 + 6x_2 &= 52 \end{aligned} \rightarrow \left[\begin{array}{cc|c} 29 & 18 & 224 \\ 3 & 25 & 162 \\ 4 & 6 & 52 \end{array} \right]$$

```
octave:2> A=[29,18,224;3,25,162;4,6,52]
A =
```

```
29    18    224
 3    25    162
 4     6     52
```

```
octave:3> rref(A)
ans =
```

```
1  0  4
0  1  6
0  0  0
```

$x_1 = 4$ bags
 $x_2 = 6$ bags