

Sec 2.2 Span

In words, the span of a set of given vectors is the set of all of the vectors that we can obtain by taking linear combinations of the given vectors.

Example: given the set of vectors $\vec{u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ show that one can reach any arbitrary vector in $\begin{bmatrix} a \\ b \end{bmatrix} \in R^2$

That is show that $\text{span}\{\vec{u}, \vec{v}\}$ is all of R^2

What is the span of the vectors $\vec{u} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$, $\vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$?

Formal Definition: The span of a set of vectors $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\} \in R^n$ denoted $\text{span}\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\}$ is the set of all possible linear combinations

$$c_1 \vec{u}_1 + c_2 \vec{u}_2 + \dots + c_m \vec{u}_m$$

of those vectors, where c_i is any real number.

Sec 2.2 Span

Example: Show that

$$\begin{bmatrix} 2 \\ 5 \\ 11 \end{bmatrix} \notin \text{span} \left\{ \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \right\}$$

Suppose $\begin{bmatrix} 2 \\ 5 \\ 11 \end{bmatrix}$ was in the span. That would mean there would exist some x_1, x_2 such that:

$A := \text{matrix}(3, 3, [4, 1, 2, -2, -1, 5, 1, 3, 11])$

$$\begin{bmatrix} 4 & 1 & 2 \\ -2 & -1 & 5 \\ 1 & 3 & 11 \end{bmatrix}$$

$\text{gausselim}(A)$

$$\begin{bmatrix} 4 & 1 & 2 \\ 0 & -\frac{1}{2} & 6 \\ 0 & 0 & \frac{87}{2} \end{bmatrix}$$

Sec 2.2 Span

Example: Show that

$$\begin{bmatrix} 5 \\ -1 \\ -7 \end{bmatrix} \in \text{span} \left\{ \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \right\}$$

Again here we seek x_1 and x_2 such that

$$x_1 \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ -7 \end{bmatrix}$$

$$\begin{aligned} 4x_1 + x_2 &= 5 \\ -2x_1 - x_2 &= -1 \\ x_1 + 3x_2 &= -7 \end{aligned}$$

$$\left[\begin{array}{cc|c} 4 & 1 & 5 \\ -2 & -1 & -1 \\ 1 & 3 & -7 \end{array} \right]$$

$A := \text{matrix}(3, 3, [4, 1, 5, -2, -1, -1, 1, 3, -7])$

$$\begin{bmatrix} 4 & 1 & 5 \\ -2 & -1 & -1 \\ 1 & 3 & -7 \end{bmatrix}$$

$\text{gausselim}(A)$

$$\begin{bmatrix} 4 & 1 & 5 \\ 0 & -\frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 0 \end{bmatrix}$$

Theorem 2.6: A vector $\vec{v} \in \text{span}\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\}$ with $\vec{v}, \vec{u}_i \in R^n$ if and only if the augmented matrix

$$\left[\vec{u}_1 \quad \vec{u}_2 \quad \cdots \quad \vec{u}_m \mid \vec{v} \right]$$

has a solution.

Proof: Just write it in it's equivalent vector form!

Sec 2.2 Span

Example find $\text{span}\left\{\begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ -3 \\ 4 \end{bmatrix}\right\}$

First, let's see if we can get to any arbitrary vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in R^3$

If so then we should be able to find a solution to the system

$$x_1 \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -3 \\ 4 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

In terms of a, b, c which places no restriction on a, b , or c . Let's make an augmented matrix and ask maple to do the heavy lifting!

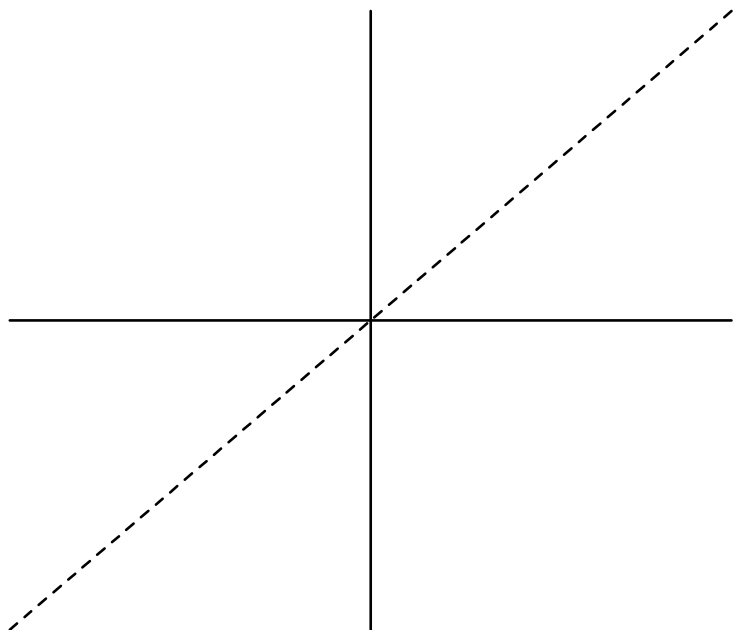
$A := \text{matrix}(3, 4, [4, 1, 5, a, -2, -1, -3, b, 1, 3, 4, c])$

$\text{gausselim}(A)$

$$\begin{bmatrix} 4 & 1 & 5 & a \\ -2 & -1 & -3 & b \\ 1 & 3 & 4 & c \end{bmatrix}$$

$$\begin{bmatrix} 4 & 1 & 5 & a \\ 0 & -\frac{1}{2} & -\frac{1}{2} & b + \frac{1}{2}a \\ 0 & 0 & 0 & c + \frac{5}{2}a + \frac{11}{2}b \end{bmatrix}$$

What does the bottom row say about the vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$?

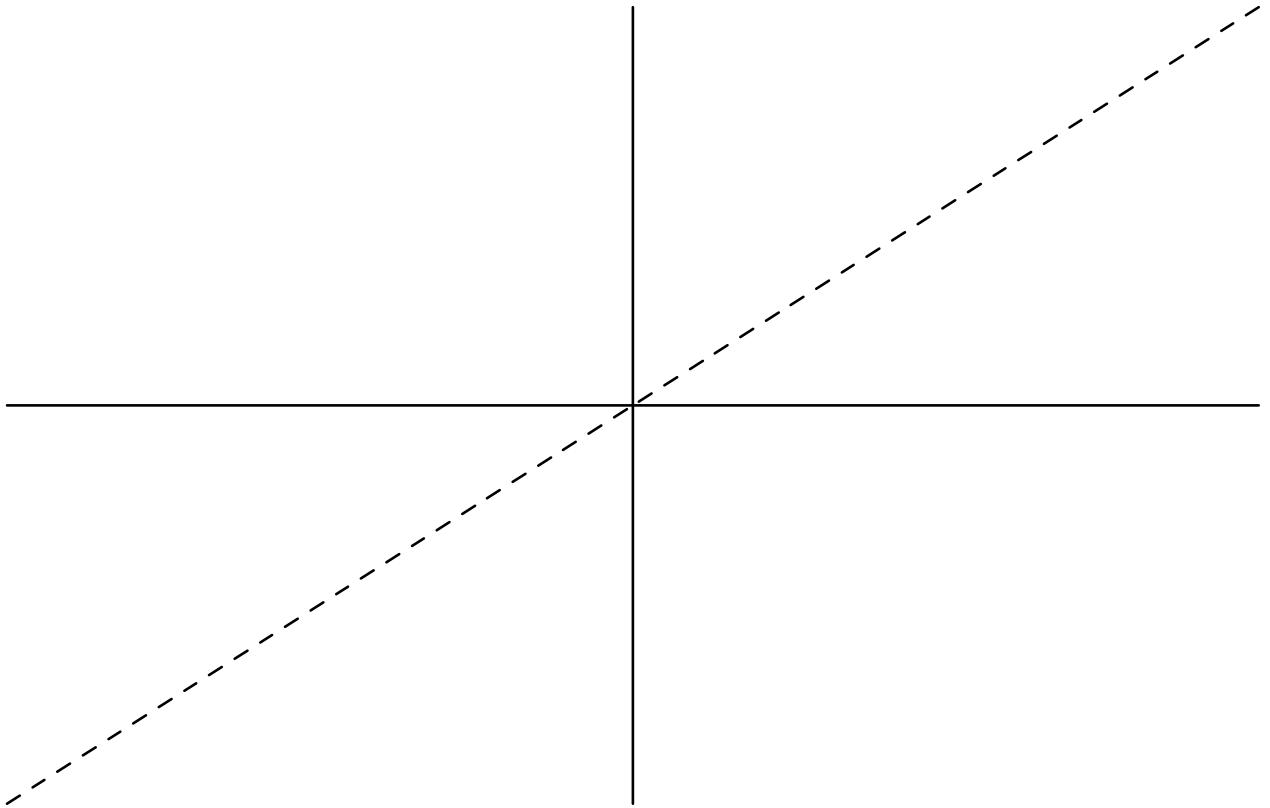


Sec 2.2 Span

We might also notice something else about this set of vectors....

Since $\begin{bmatrix} 5 \\ -3 \\ 4 \end{bmatrix} \in \text{span} \left\{ \begin{bmatrix} 4 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \right\}$ we only need to look at the two of the vectors in the list.

Let's think about the span of two vectors, which aren't multiples of each other, in R^3



Sec 2.2 Span

3 vectors that do span all of R^3

$$\text{Let } \vec{u}_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}, \vec{u}_2 = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}$$

$$x_1 \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$A := \text{matrix}(3, 4, [2, -1, 3, a, 1, -2, 1, b, -1, 2, 1, c]);$$

$$\begin{bmatrix} 2 & -1 & 3 & a \\ 1 & -2 & 1 & b \\ -1 & 2 & 1 & c \end{bmatrix}$$

$$\text{gausselim}(A)$$

$$\begin{bmatrix} 2 & -1 & 3 & a \\ 0 & -\frac{3}{2} & -\frac{1}{2} & b - \frac{1}{2}a \\ 0 & 0 & 2 & c + b \end{bmatrix}$$

Here we had 3 vectors in R^3 that span all of R^3 and we saw in the previous example that two vectors in R^3 can't span the R^3 , but they could span plane or R^2 . It is also clear that no combination vectors in R^2 could ever span R^3 , they don't have enough components. This leads us to the following theorem

Theorem 2.9: Let $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\} \in R^n$ if $m < n$ then the vectors cannot span R^n if $n \leq m$ then they might span R^n , however this will depend on the vectors.

Proof:

Because there are more rows than columns in the augmented system our echelon form will necessarily have one or more row of zeros to the left of the dividing line. To show that we can't reach every vector in R^n we only need to choose a vector such that when we apply our row operations to go to echelon form we get a non zero value in the n th spot. In the picture below that means $v'_n \neq 0$ then we have a no solution case! This vector could be easily found by choosing $\vec{v}' = (0, 0, \dots, 1)$.

$$\left[\begin{array}{cccc|c} u_{11} & u_{12} & \cdots & u_{1m} & v_1 \\ u_{21} & u_{22} & \cdots & u_{2m} & v_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ u_{n1} & u_{n2} & \cdots & u_{nm} & v_n \end{array} \right]$$

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1m} & v'_1 \\ 0 & a_{22} & \cdots & a_{2m} & v'_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & v'_n \end{array} \right]$$

Sec 2.2 Span

To illustrate:

Show the set of vectors $\left\{ \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \right\}$ do not span all of R^3

Of course a natural question to ask is if $m \geq n$ how can we know when $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\} \in R^n$ will span all of R^n . We have a theorem for that too!

Theorem 2.8: The set of vectors $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\} \in R^n$ will span R^n if and only if the echelon form of the matrix formed with the vectors as columns

$$A = [\vec{u}_1 \quad \vec{u}_2 \quad \cdots \quad \vec{u}_n]$$

has a pivot position in every row

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & v'_1 \\ 0 & a_{22} & \cdots & a_{2n} & v'_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{nn} & v'_n \end{array} \right]$$

if $n = m$

The no solution case is impossible!

$$\left[\begin{array}{ccccc|c} a_{11} & a_{12} & \cdots & a_{1m-1} & a_{1m} & v'_1 \\ 0 & a_{22} & \cdots & a_{2m-1} & a_{2m} & v'_2 \\ \vdots & \vdots & \square & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & a_{nm-1} & a_{nm} & v'_n \end{array} \right]$$

if $n < m$

Sec 2.2 Span

The equation $A\vec{x} = \vec{b}$

Let $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_m$ be vectors in R^n . If

$$A = [\vec{a}_1 \ \vec{a}_2 \ \cdots \ \vec{a}_m] \text{ and } \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

Then

$$A\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_m\vec{a}_m = \vec{b}$$

Example calculate:

$$A\vec{x} = \begin{bmatrix} 1 & 0 & 1 \\ -2 & 3 & -1 \\ 4 & 1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} = \vec{b}$$

Theorem 2.11: $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_m$ be vectors in R^n . Then the following statements are equivalent. That is, if one is true, then so are the others, and if one is false, then so are the others.

1. $\vec{b}$ is in the span of $\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_m\}$.
2. The vector equation $x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_m\vec{a}_m = \vec{b}$ has at least one solution
3. The system corresponding to $[\vec{a}_1 \ \vec{a}_2 \ \cdots \ \vec{a}_m \mid \vec{b}]$ has at least one solution
4. The equation $A\vec{x} = \vec{b}$, with A having $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_m$ as its columns and $\vec{x}, \vec{b} \in R^m$ has at least one solution