

Section 2.3 Linear Independence

Linear independence is an important concept in linear algebra and sets of vectors which are linearly independent allow us to guarantee useful properties about those vectors. A set of vectors is linearly independent when no one vector in the set can be written as a linear combination of the others. In a sense it means it is not redundantly listed in the set. We can state this equivalently in the following way.

Let $\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \dots, \vec{u}_m\}$ be a set of vectors in R^n if the only solution to

$$x_1\vec{u}_1 + x_2\vec{u}_2 + x_3\vec{u}_3 + \dots + x_m\vec{u}_m = 0$$

is the trivial solution, $x_1 = x_2 = x_3 = \dots, x_m = 0$ then the set of vectors is **linearly independent**, if there are any non trivial (non-zero) solutions the set is called **linearly dependent**.

Example: Determine if the following set of vectors are linearly independent

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 5 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Let's think about what this would mean in terms of a linear system.

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```
octave:32> u1=[1;2;-3;1]
u1 =
```

```
1
2
-3
1
```

```
octave:34> u3=[1;3;5;-1]
u3 =
```

```
1
3
5
-1
```

```
octave:33> u2=[4;-2;1;0]
u2 =
```

```
4
-2
1
0
```

```
octave:35> u4=[0;2;1;1]
u4 =
```

```
0
2
1
1
```

```
octave:36> A=[u1,u2,u3,u4,zeros(4,1)]
A =
```

```
1    4    1    0    0
2   -2    3    2    0
-3    1    5    1    0
1    0   -1    1    0
```

```
octave:37> rref(A)
ans =
```

```
1    0    0    0   -0
0    1    0    0    0
0    0    1    0    0
0    0    0    1    0
```

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Your turn! Grab your laptops and head over to octave-online.net (5 min in class excise)

Determine if the following set of vectors is linearly independent

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 5 \\ -1 \end{bmatrix}, \begin{bmatrix} 10 \\ -14 \\ -23 \\ 6 \end{bmatrix} \right\}$$

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Show that the vectors

$$\left\{ \begin{bmatrix} 16 \\ 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 22 \\ 4 \\ 4 \end{bmatrix}, \begin{bmatrix} 18 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 18 \\ 2 \\ 6 \end{bmatrix} \right\}$$

are linearly dependent

```
octave:4> u1=[16;2;8]
```

```
u1 =
```

```
16  
2  
8
```

```
octave:8> A=[u1,u2,u3,u4,zeros(3,1)]
```

```
A =
```

```
16    22    18    18     0  
2     4     0     2     0  
8     4     4     6     0
```

```
octave:5> u2=[22;4;4]
```

```
u2 =
```

```
22  
4  
4
```

```
octave:11> rref(A)
```

```
ans =
```

```
1     0     0     1/2     0  
0     1     0     1/4    -0  
0     0     1     1/4    -0
```

```
octave:6> u3=[18;0;4]
```

```
u3 =
```

```
18  
0  
4
```

So we have nonzero solutions!

```
octave:7> u4=[18;2;6]
```

```
u4 =
```

```
18  
2  
6
```

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Consider the set of vectors $\left\{ \begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}, \begin{bmatrix} e \\ f \end{bmatrix} \right\}$ where a, b, c, d, e, f can be any real numbers we like. Do you think it is possible to find values for a, b, c, d, e, f such that these vectors are linearly independent?

Case 1: At least one of the vectors is a multiple of another.

Case 2 None of the vectors are multiples of each other.

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Theorem 2.14: For a set of vectors $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\}$ in R^n , if $n < m$ they are linearly dependent.

This is because a system of homogeneous equations with more variables than equations must have infinitely many solutions .

Quick Definition a homogeneous equation is one where the sum of all nonzero terms add up to zero, a homogeneous system is a system of equations made up of homogeneous equations.

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n & = & 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n & = & 0 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + \dots + a_{3n}x_n & = & 0 \\ \vdots & & \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n & = & 0 \end{array}$$

To see why a homogeneous system with less equations than variables we only need to think about the possible echelon forms of such a system.

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n & = & 0 \\ a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n & = & 0 \\ a_{33}x_3 + \dots + a_{3n}x_n & = & 0 \\ \vdots & & \vdots \\ a_{mn-1}x_{n-1} + a_{mn}x_n & = & 0 \end{array}$$

Either we have a free variable coming from a missing pivot position, this is possible too for non-homogeneous systems

$$\begin{array}{rcl} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n & = & 0 \\ a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n & = & 0 \\ a_{33}x_3 + \dots + a_{3n}x_n & = & 0 \\ \vdots & & \vdots \\ 0 & = & 0 \end{array}$$

Or one of the left hand sides was a multiple of another, but since all the right hand sides are zero its $0=0$ giving a free variable. The right hand sides being zero makes it homogeneous and it is that feature that guarantees us a free variable in the case where a left hand side was a multiple of another.

With that in mind, if $n < m$, we have n equations coming from the n components of each vector and m variables coming from each needed multiple of the m vectors. So we have more unknowns than equations in a homogeneous system, therefore there are infinitely many solutions one of which is not trivial and the vectors must be linearly dependent.

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Theorem 2.15: Let $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\}$ be a set of vectors in R^n . Then this set is linearly dependent if and only if one of the vectors in the set is in the span of the other vectors.

Proof. Suppose the set is linearly independent

Suppose one vector is in the span of the others

Theorem 2.16: Let $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\} \in R^n$, and suppose we make a matrix with the vectors as columns

$$A = [\vec{u}_1 \ \vec{u}_2 \ \dots \ \vec{u}_n]$$

When the matrix A is transformed into matrix B with row operations with B in echelon form then

- (1) $\text{span}\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\} = R^n$ exactly when B has a pivot in every row.
- (2) $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\}$ is linearly independent exactly when B has a pivot position in every column

$$\begin{bmatrix} a & b & c & d \\ 0 & e & f & g \\ 0 & 0 & h & i \end{bmatrix}$$

Pivot in every
row only

Showed this with theorem 2.8. Quick reasoning is that if there is a pivot position in every row there is no row that can look like $0 = q$ where $q \neq 0$ meaning we won't have a no solution case if we augment the system with any vector from R^n . Thus we could always find a solution to the augmented system with any vector from R^n

$$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \\ 0 & 0 & 0 \end{bmatrix}$$

Pivot in every
column only

In this case, because there is a pivot in every column, one row must be all zeros except the m^{th} variable position. This means that when the system is augmented with the zero vector the m^{th} variable must be zero. Since there is also a pivot position in all other columns there can be no free parameters for the system which means they must all be zero as well. Thus only the trivial solution would exist

$$\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$$

Pivot in every
Row and column

This case is a combination of the previous 2 so we get them both! Span and linear independence.

In this case we would say that the original columns of A form a basis for R^n . We will talk a lot about the idea of a basis.

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Theorem 2.16: Let $A = [\vec{u}_1 \ \vec{u}_2 \ \cdots \ \vec{u}_n]$ and $\vec{x} = (x_1, x_2, \dots, x_m)$. The set $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\}$ is linearly independent if and only if the homogeneous linear system:

$$A\vec{x} = 0$$

Has only the trivial ($\vec{x} = 0$) solution.

If we multiply the system out

We see it is the same as the definition of linear independence

Theorem 2.19: Let $\vec{x}_p$ be a particular solution of the system

$$A\vec{x} = b$$

And let $\vec{x}_h$ be the homogeneous solution to the system

$$A\vec{x} = 0$$

Then the vector $\vec{x}_g = \vec{x}_p + \vec{x}_h$ also solves

$$A\vec{x} = b$$

Just a small note on why this is important. In practice the system $A\vec{x} = b$ may model something physical and the vector $\vec{x}_h$ may not be the zero vector. In that case we want to include it in our solution. So often when we solve $A\vec{x} = b$ we also must solve $A\vec{x} = 0$. This is especially true in differential equations.

Proof:

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A Unifying Theorem. 2.21 Let $S = \{\vec{u}_1, \dots, \vec{u}_n\}$ be a set of n vectors in R^n , and let $A = [\vec{u}_1 \ \vec{u}_2 \ \dots \ \vec{u}_n]$. Then the following are equivalent.

- (1) S spans R^n
- (2) S is linearly independent
- (3) $A\vec{x} = \vec{b}$ has a unique solution for all $\vec{b}$ in R^n

For this we need to prove that (1) $\Rightarrow$ (2), (2) $\Rightarrow$ (3), (3) $\Rightarrow$ (1)

The 123 circle....

① $\Rightarrow$ 2
② $\Rightarrow$ 3
③ $\Rightarrow$ 1

Theorem 2.16: Let $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\} \in R^n$, and suppose we make a matrix with the vectors as columns

$$A = [\vec{u}_1 \ \vec{u}_2 \ \dots \ \vec{u}_m]$$

When the matrix A is transformed into matrix B with row operations with B in echelon form then

- (1) $\text{span}\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\} = R^n$ exactly when B has a pivot in every row.
- (2) $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\}$ is linearly independent exactly when B has a pivot position in every column

Formal Definition: The span of a set of vectors $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\} \in R^n$ denoted $\text{span}\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\}$ is the set of all possible linear combinations

$$c_1\vec{u}_1 + c_2\vec{u}_2 + \dots + c_m\vec{u}_m$$

of those vectors, where c_i is any real number.

(1) $\Rightarrow$ (2)

If $\text{span}\{S\} = R^n$ by theorem 2.16 the matrix $[\vec{u}_1 \ \vec{u}_2 \ \dots \ \vec{u}_n]$ since there are n rows and n columns, this means there must also be a pivot in every column, thus also by theorem 2.18 the set S is made of linearly independent vectors.

(2) $\Rightarrow$ (3) If the set S is linearly independent again by theorem 2.16 the matrix $[\vec{u}_1 \ \vec{u}_2 \ \dots \ \vec{u}_n]$ has a pivot position in every column, but there are also only n rows so there is also a pivot position in every row. This means it is impossible to have a row of zeros to the left of the $|$ in any augmented matrix of the form

$[\vec{u}_1 \ \vec{u}_2 \ \dots \ \vec{u}_n | \vec{b}]$ for any arbitrary $\vec{b} \in R^n$ we may put to the right of the dividing line. This means a $0 = q$ or $0 = 0$ case is impossible, outlawing any case of no solution or free variables. Thus one must be able to back solve for a unique solution.

(3) $\Rightarrow$ (1)

If we simply write $A\vec{x} = \vec{b}$ in its equivalent vector equation form:

$$x_1\vec{u}_1 + x_2\vec{u}_2 + \dots + x_n\vec{u}_n = \vec{b}$$

We see that because we are supposed there is a solution to $A\vec{x} = \vec{b}$ we know that there exist $x_1 \dots x_n$ which form a linear combination of the $\vec{u}$'s to get to $\vec{b}$ for any arbitrary $\vec{b} \in R^n$. Thus the set S spans R^n by definition of span.

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Now your turn! Show that all (a)-(d) are equivalent

Theorem 2.20: Let $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\}, \vec{b}$ be vectors in R^n . Then the following statements are equivalent. This means they if one is true they all are and if one is false they all are.

- (a) The set $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\}$ is linearly independent
- (b) The vector equation $x_1\vec{u}_1 + x_2\vec{u}_2 + \dots + x_m\vec{u}_m = \vec{b}$ has at most one solution for every $\vec{b}$.
- (c) The linear system corresponding to $[\vec{u}_1 \ \vec{u}_2 \ \dots \ \vec{u}_m \ | \ \vec{b}]$ has at most one solution for every $\vec{b}$
- (d) The equation $A\vec{x} = \vec{b}$, with $A = [\vec{u}_1 \ \vec{u}_2 \ \dots \ \vec{u}_m]$ has at most one solution for every $\vec{b}$

Hint: $b \Rightarrow c, c \Rightarrow d$ is easy, those are all equivalent statements of the same equation, if one has a solution they all do! So now we just need to argue that $a \Rightarrow b$ and $d \Rightarrow a$