

Section 3.1 Linear Transformations

A linear transformation is a very important concept in linear algebra and can be used to represent a wide variety of physical, economic, statistical and scientific problems. In this section we will study linear transforms that arise from the multiplication of a vector by a Matrix. Geometrically the action of the matrix on the vectors will be to shrink, stretch and reorient them.

A function $T: R^m \rightarrow R^n$ is a **linear transformation** if for all vectors $\vec{u}, \vec{v} \in R^m$ and all scalars r we have the following properties.

(a) $T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$

(b) $T(r\vec{u}) = rT(\vec{u})$

The properties given in (a) and (b) above are extremely useful in practice. Other things are often called linear as well and there is a whole theory on Linear Operators. The differentiation operator is one such operator. If you recall the sum of the derivatives is the derivative of the sum rule, that's (a). You might also remember the fact that when taking derivatives you can pull out a constant multiple, that's just property (b).

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x) \quad \frac{d}{dx}cf(x) = c\frac{d}{dx}f(x)$$

Examples of transformations, are they linear?

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x$$

$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}, \begin{bmatrix} x_1+x_2 \\ y_1+y_2 \end{bmatrix}, r\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = \begin{bmatrix} rx_1 \\ ry_1 \end{bmatrix}$
 $T\left(\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right)$
 $x_1+x_2 = x_1 + x_2$
 $T\left(r\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) = rT\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right)$
 $r x_1 = x_1$
 $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} y+x \\ z-y \\ x+y+z \end{bmatrix}$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos(x) \\ \sin(x) \end{bmatrix}$$

$T\left(\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right)$
 $\begin{bmatrix} \cos(x_1+x_2) \\ \sin(y_1+y_2) \end{bmatrix} \neq \begin{bmatrix} \cos(x_1) \\ \sin(y_1) \end{bmatrix} + \begin{bmatrix} \cos(x_2) \\ \sin(y_2) \end{bmatrix}$
 Not Linear

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x+1 \\ xy \end{bmatrix}$$

$T\left(\begin{bmatrix} x_1+y_1 \\ x_2+y_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right)$
 $\begin{bmatrix} 2(x_1+y_1)+1 \\ (x_1+y_1)(x_2+y_2) \end{bmatrix} \neq \begin{bmatrix} 2x_1+1 \\ x_1y_1 \end{bmatrix} + \begin{bmatrix} 2x_2+1 \\ x_2y_2 \end{bmatrix}$
 NO

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} y^2 \\ x^2 \\ z \end{bmatrix}$$

$(x_1+y_1)^2 \neq x_1^2 + y_1^2$

NO

$T\left(\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \\ z_1+z_2 \end{bmatrix}\right) = T\left(\begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}\right)$
 $\begin{bmatrix} y_1+y_2+x_1+x_2 \\ z_1+z_2-(y_1+y_2) \\ x_1+x_2+y_1+y_2+z_1+z_2 \end{bmatrix} = \begin{bmatrix} y_1+x_1 & + & y_2+x_2 \\ z_1-y_1 & + & z_2-y_2 \\ x_1+y_1+z_1 & + & x_2+y_2+z_2 \end{bmatrix}$

Section 3.1 Linear Transformations

Linear Transformations as matrix multiplication

Let's revisit the linear transformation $T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + y \\ y - 3z \\ -x \\ x + y + z \end{bmatrix}$, I claim we can write this in an equivalent form using a matrix times a vector. That is $T(\vec{u}) = A\vec{u}$.

Let $\vec{a}_1, \vec{a}_2, \dots, \vec{a}_m$ be vectors in R^n . If

$$A = [\vec{a}_1 \ \vec{a}_2 \ \dots \ \vec{a}_m] \text{ and } \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \text{ Then}$$

$$A\vec{x} = \vec{x}_1\vec{a}_1 + \vec{x}_2\vec{a}_2 + \dots + \vec{x}_m\vec{a}_m = \vec{b}$$

$$x \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + z \begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2x \\ 0 \\ -x \\ x \end{bmatrix} + \begin{bmatrix} y \\ y \\ 0 \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -3z \\ 0 \\ z \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -3 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} =$$

Here in one line is the correspondence between the forms of our linear transformation.

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x + y \\ y - 3z \\ -x \\ x + y + z \end{bmatrix} = \begin{bmatrix} 2x \\ 0 \\ -x \\ x \end{bmatrix} + \begin{bmatrix} y \\ y \\ 0 \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ -3z \\ 0 \\ z \end{bmatrix} = x \begin{bmatrix} 2 \\ 0 \\ -1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + z \begin{bmatrix} 0 \\ -3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 1 & -3 \\ -1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Theorem 3.2: Let A be an $n \times m$ matrix and $\vec{u} \in R^m$. Then $T(\vec{u}) = A\vec{u}$ is a linear transformation $T: R^n \rightarrow R^m$.

Proof

$$T(u_1 + u_2) = A(\vec{u}_1 + \vec{u}_2) = A\vec{u}_1 + A\vec{u}_2 \quad \checkmark = T(\vec{u}_1) + T(\vec{u}_2)$$

$$T(r\vec{u}_1) = rT(\vec{u}_1)$$

$$A(r\vec{u}) = rA\vec{u} \quad \checkmark$$

Section 3.1 Linear Transformations

Let $T: \mathbb{R}^m \rightarrow \mathbb{R}^n$ be a linear transformation $T(\vec{u}) = \vec{v}$.

We call the set of input vectors $\{\vec{u}\} \in \mathbb{R}^m$ the **Domain** of T

We call the set of output vectors $\{\vec{v}\} \in \mathbb{R}^n$ the **Codomain** of T

For a specific vector $\vec{u}$ in the domain of T we call $\vec{v} = T(\vec{u})$ the **Image** of $\vec{u}$ under T

The collection of the images of all of the vectors in the domain is called the **Range** of T

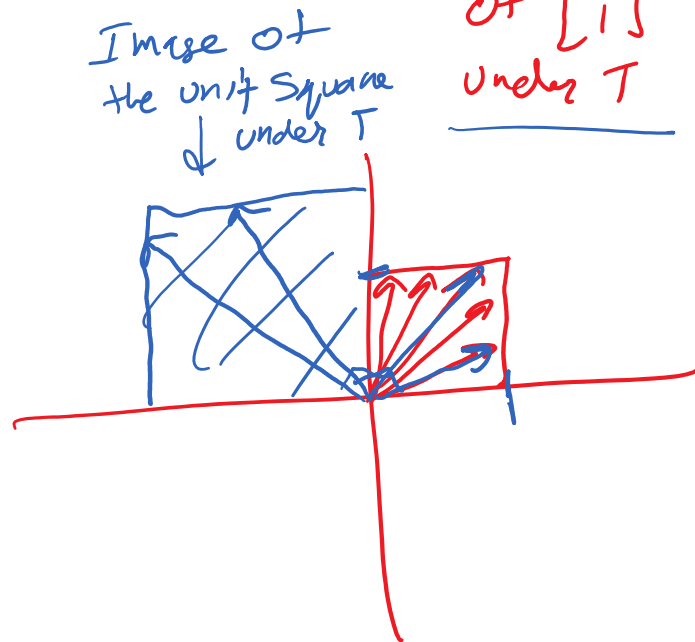
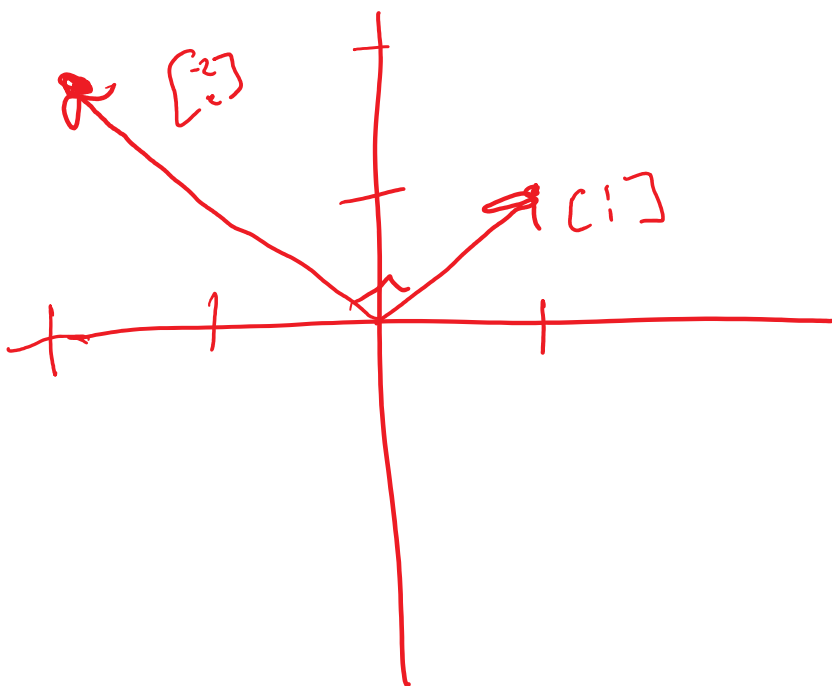
Example:

Define $T(\vec{u}) = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$ Identify the Domain and Codomain and draw the Image of the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. What does the image of all of the vectors in the unit square $[0,1] \times [0,1]$ look like?

Domain: $\mathbb{R}^2$ Codomain: $\mathbb{R}^2$

$$T\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 0 \\ 2 \end{bmatrix} + 1 \begin{bmatrix} -2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$

Image of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ under T



Section 3.1 Linear Transformations

Theorem 3.3: Let $A = [\vec{a}_1 \ \vec{a}_2 \ \cdots \ \vec{a}_m]$ be a $n \times m$ matrix, and let $T: R^m \rightarrow R^n$ with $T(\vec{x}) = A\vec{x}$ be a linear transformation. Then

- (a) The vector $\vec{w}$ is in the range of T if and only if $A\vec{x} = \vec{w}$ is a consistent linear system
 (b) $\text{range}(T) = \text{span}\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_m\}$

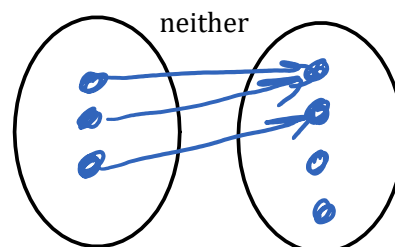
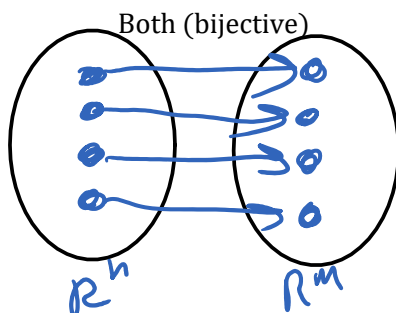
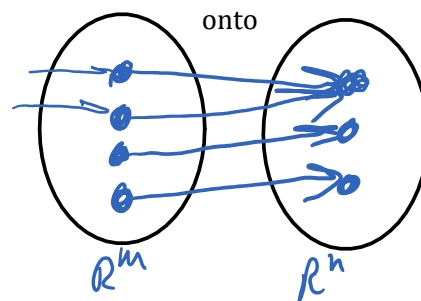
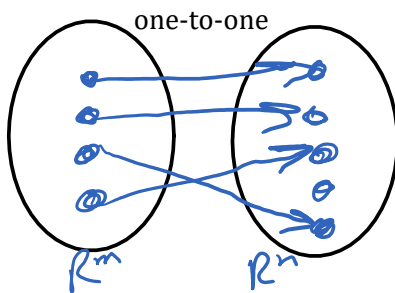
Proof of (b)

$$A\vec{x} = [\vec{a}_1 \ \vec{a}_2 \ \cdots \ \vec{a}_m] \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$= x_1\vec{a}_1 + x_2\vec{a}_2 + x_3\vec{a}_3 + \cdots + x_n\vec{a}_n = \text{span}\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_m\}$$

Let $T: R^m \rightarrow R^n$ be a linear transformation then

- (a) T is one-to-one (injective) if for every vector $\vec{w} \in R^n$ there exists at most one vector $\vec{u} \in R^m$ such that $T(\vec{u}) = \vec{w}$.
 (b) T is onto (surjective) if for every vector $\vec{w} \in R^n$ there exists at least one vector $\vec{u} \in R^m$ such that $T(\vec{u}) = \vec{w}$.



Section 3.1 Linear Transformations

Example: Assign the correct terminology to the linear transformations

$$T: \mathbb{R}^2 \rightarrow \mathbb{R}^1$$

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x$$

1-1? NO

$$T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = 1$$

$$T\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) = 1$$

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}$$

1-1 x ≠ y YES
2x ≠ 2y

onto YES!

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x - y \\ 2y - z \\ 2z - y \end{bmatrix}$$

$$\begin{aligned} 2x - y &= 0 \\ 2y - z &= 0 \\ 2z - y &= 0 \end{aligned}$$

$$T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = \begin{bmatrix} 1 & 1 & 1 & -3 \\ -1 & 0 & -4 & 5 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$$

This one is a bit harder to tell off the bat, but we have a theorem that is helpful.

Theorem 3.5: Let T be a linear transformation. Then T is one-to-one if and only if the only solution to $T(\vec{x}) = \vec{0}$ is the trivial solution $\vec{x} = \vec{0}$.

Proof:

Suppose T is 1-1

$$[\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n]$$

$$T(\vec{x}) = A\vec{x} =$$

$$\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2 + \dots + x_n\vec{a}_n = \vec{0}$$

$$\vec{x} = \vec{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$0\vec{a}_1 + 0\vec{a}_2 + \dots + 0\vec{a}_n = \vec{0}$$

Since T is 1-1 $\vec{0}$ is the only solution

Suppose the only solution to $T(\vec{x}) = \vec{0}$ is $\vec{x} = \vec{0}$. $\Rightarrow T$ 1-1

Suppose T is not 1-1

$$T(\vec{u}_1) = \vec{b} \quad T(\vec{u}_2) = \vec{b}$$

$$T(\vec{u}_1) - T(\vec{u}_2) = \vec{b} - \vec{b} = \vec{0}$$

$$T(\vec{u}_1) - T(\vec{u}_2) = \vec{0}$$

$$T(\vec{u}_1) + T(-\vec{u}_2) = \vec{0}$$

$$T(\vec{u}_1 - \vec{u}_2) = \vec{0}$$

What must $\vec{u}_1 - \vec{u}_2$ be?

$$\vec{u}_1 - \vec{u}_2 = \vec{0} \rightarrow \vec{u}_1 = \vec{u}_2$$

T is linear

$$T(\vec{u}_1) + T(\vec{u}_2) = T(\vec{u}_1 + \vec{u}_2)$$

$$= T(\vec{v}) = rT(\vec{u})$$

Section 3.1 Linear Transformations

Let A be an $n \times m$ matrix and define $T: R^m \rightarrow R^n$ by $T(\vec{x}) = A\vec{x}$. Then

- (a) T is 1-1 if and only if the columns of A are linearly independent
- (b) If $A \sim B$ where B is in echelon form then T is 1-1 if and only if B has a pivot position in every column.
row operations take $A \rightarrow B$
- (c) If $n < m$, then T is **not** 1-1
 $\text{Span}\{\vec{a}_1, \dots, \vec{a}_n\} = R^n$
- (d) T is onto if and only if the columns of A span the codomain R^n
- (e) If $A \sim B$ and B is in echelon form, then T is onto if and only if B has a pivot position in every row
- (f) If $n > m$, the T is **not** onto.

Theorem 3.9 The second unifying theorem

Let $S = \{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\}$ be a set of n vectors in R^n , Let $A = [\vec{a}_1 \cdots \vec{a}_n]$, and let $T: R^n \rightarrow R^n$ be given by $T(\vec{x}) = A\vec{x}$. Then the following are equivalent.

- (a) S spans R^n
- (b) S is linearly independent
- (c) $A\vec{x} = \vec{b}$ has a solution for all $\vec{b} \in R^n$
- (d) T is onto
- (e) T is one-to-one

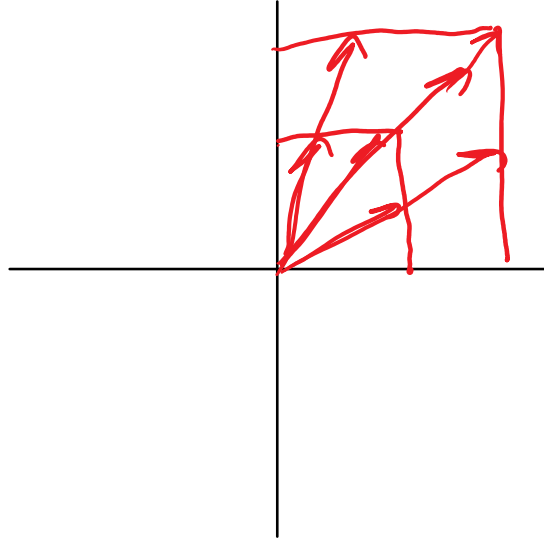
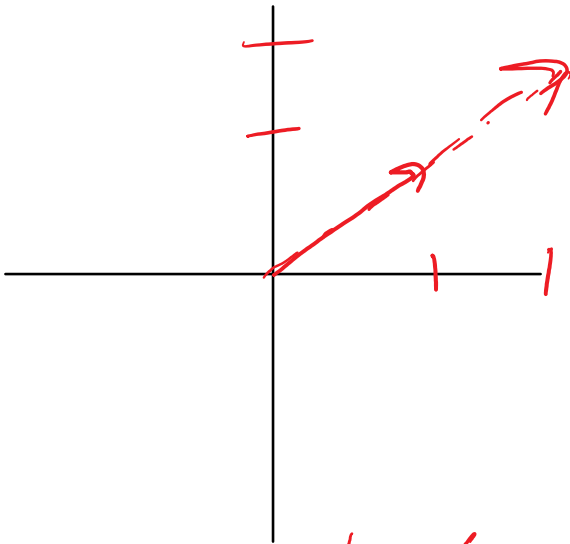
Section 3.1 Linear Transformations

Geometry of linear transformations. As we stated in the beginning linear transformations can shrink, stretch and reorient (rotate) vectors.

Scaling linear transformation $\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}$

Ex Draw the before and after of the transformation $T \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ sketch what this does to the unit square

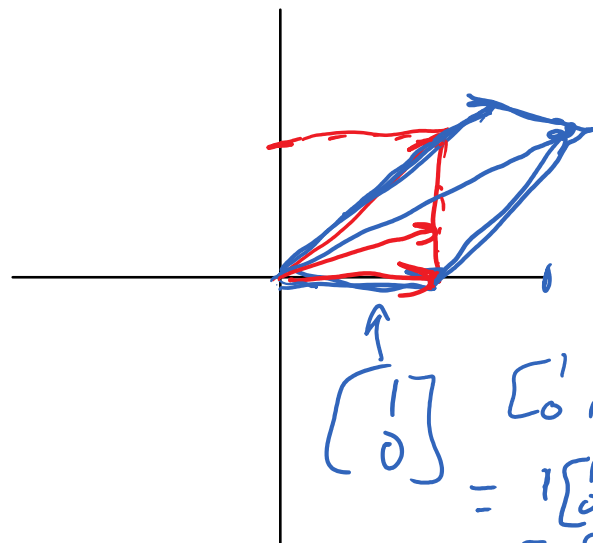
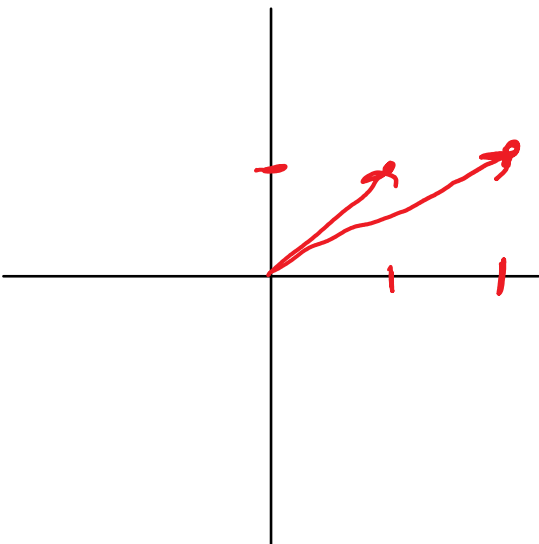
$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = (1) \begin{bmatrix} 2 \\ 0 \end{bmatrix} + (1) \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$



Shear linear transformation $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$

Ex Draw the before and after of the transformation $T \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, sketch what this does to the unit square

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$



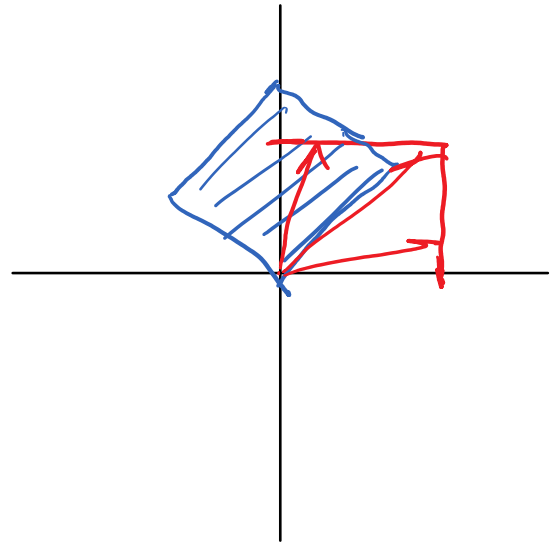
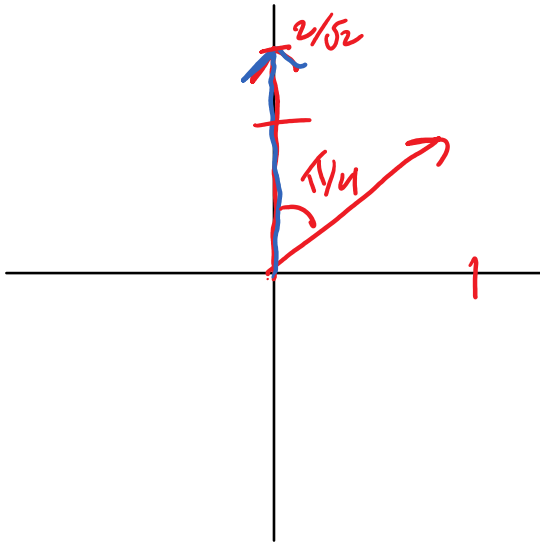
$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Section 3.1 Linear Transformations

Rotation $\begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix}$ $\cos(\alpha) = 1/\sqrt{2}$ $\sin(\alpha) = 1/\sqrt{2}$ $\alpha = \pi/4$

$$1 \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} + 1 \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 2/\sqrt{2} \end{bmatrix}$$

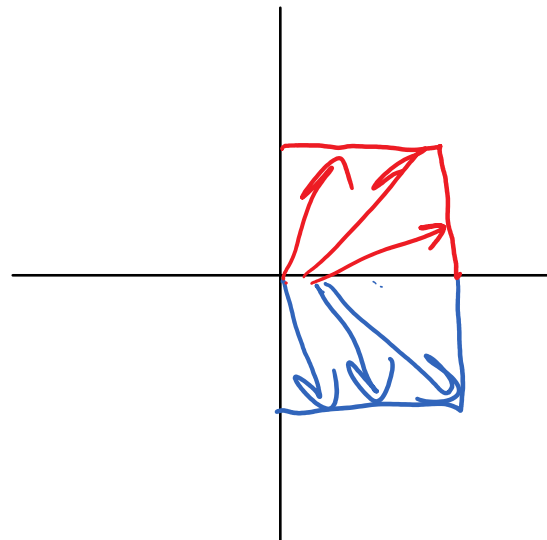
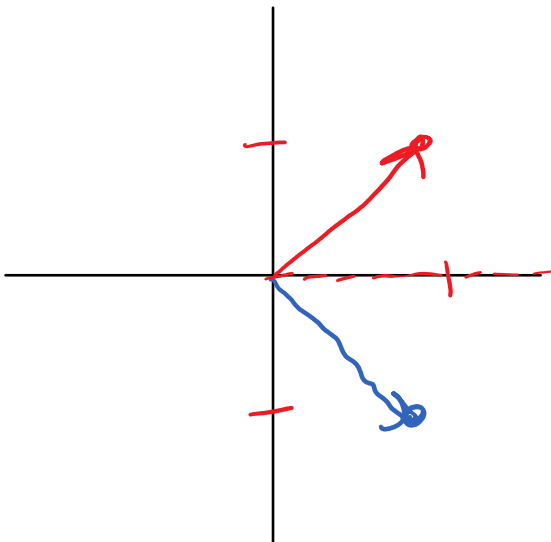
Ex Draw the before and after of the transformation $T \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ sketch what this does to the unit square



Reflection $\begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{bmatrix}$ $\alpha = 0$

$$1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Ex Draw the before and after of the transformation $T \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ sketch what this does to the unit square

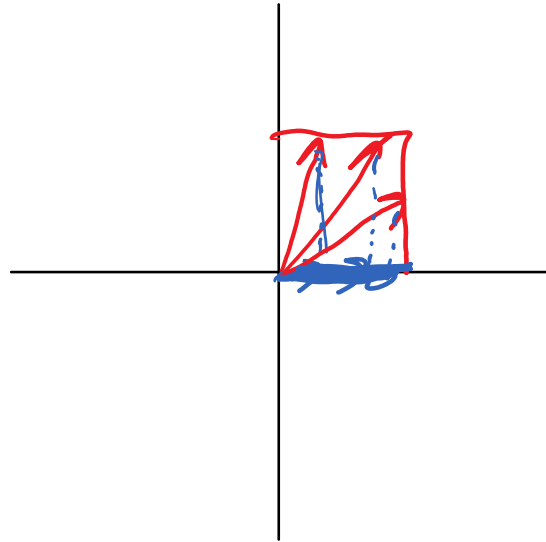
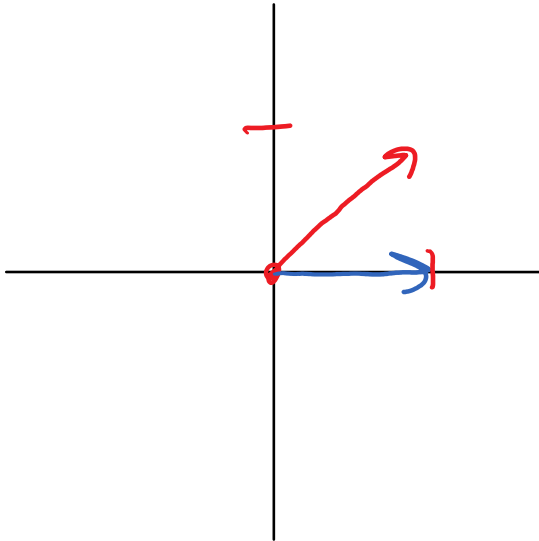


Section 3.1 Linear Transformations

Projection $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Ex Draw the before and after of the transformation $T \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, sketch what this does to the unit square.

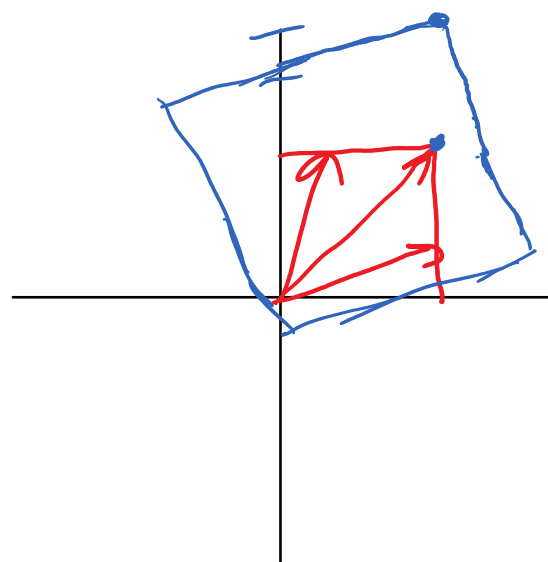
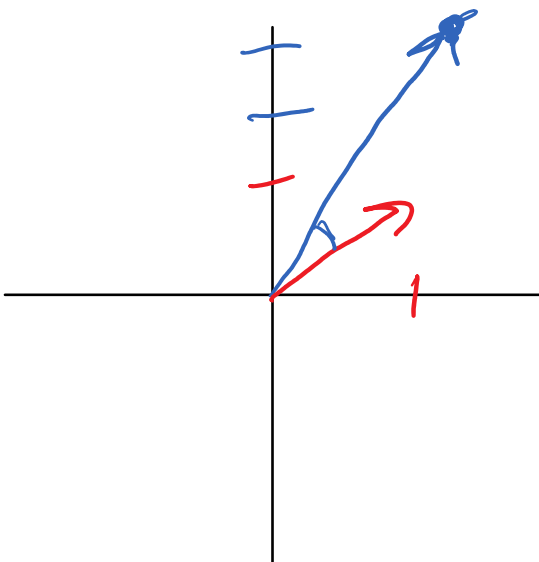
$$1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{bmatrix} 1 + 0 \\ 0 + 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$



Rotation dilatation $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

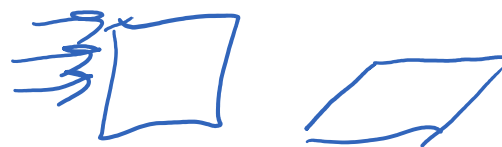
Ex Draw the before and after of the transformation $T \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, sketch what this does to the unit square

$$\rightarrow 1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$



Section 3.1 Linear Transformations

Combining linear transformations.



Ex: Find a sequence of linear transformations that rotates a vector in $\mathbb{R}^2$ by 90° , scales it by 2, and shears it.

Rotation

$$\begin{bmatrix} \cos(\alpha) & -\sin(\alpha) \\ \sin(\alpha) & \cos(\alpha) \end{bmatrix} \rightarrow \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$90^\circ \pi/2$

$$\begin{bmatrix} \cos(\pi/2) & -\sin(\pi/2) \\ \sin(\pi/2) & \cos(\pi/2) \end{bmatrix}$$

Scaling

$$\begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix} \rightarrow \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

Shear

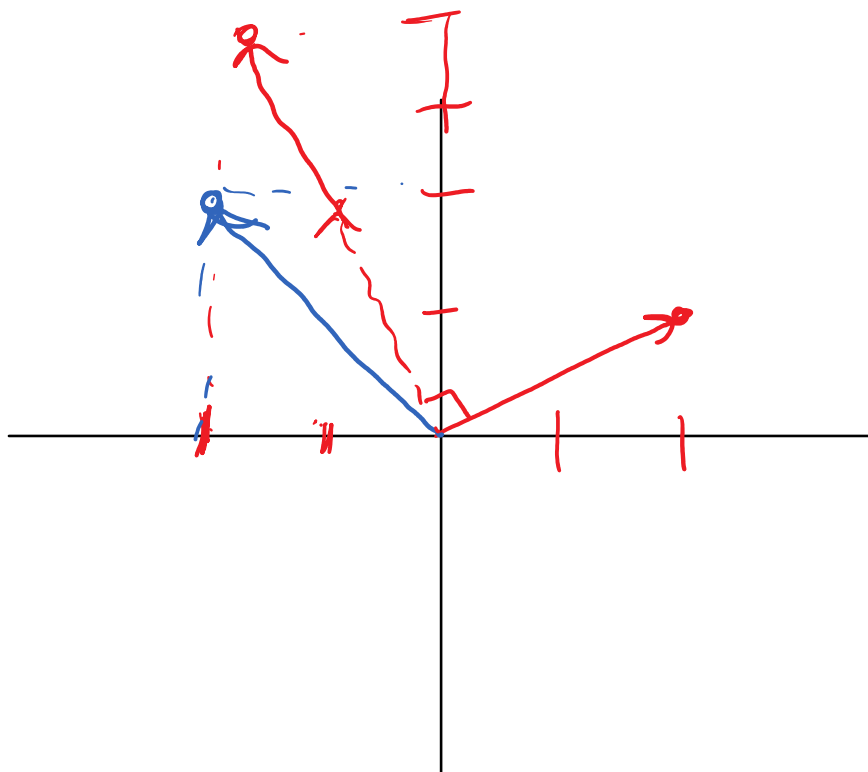
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Apply the transformation to the vector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and sketch it and its image.

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

$$2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} -2+0 \\ -2+4 \end{bmatrix} = \begin{bmatrix} -2 \\ 2 \end{bmatrix}$$



Section 3.1 Linear Transformations

Show that if $T(x+y) = T(x) + T(y)$ and $T(cx) = cT(x)$ (that is, T is linear) then

$$T(c_1x + c_2y) = c_1T(x) + c_2T(y)$$

$$T(c_1x + c_2y) = T(c_1x) + T(c_2y) \quad \text{Start here}$$

$$\text{Since } T(x+y) = T(x) + T(y) \leftarrow \text{know given } \textcircled{1}$$

$$\text{and } T(c_1x) = c_1T(x), T(c_2x) = c_2T(x) \leftarrow \text{know given } \textcircled{2}$$

$$\text{since } T(cx) = cT(x)$$

Thus

$$\underline{T(c_1x + c_2y)} \stackrel{\textcircled{1}}{=} \underline{T(c_1x)} + \underline{T(c_2y)} \stackrel{\textcircled{2}}{=} \underline{c_1T(x) + c_2T(y)} \quad \square$$

$$T(\dot{\cdot}) + (\dot{\cdot}) = T(\dot{\cdot}) + T(\dot{\cdot})$$