

Section 3.2 Matrix Algebra

Matrices are mathematical objects and we can do fun things like add them and multiply them. The set of $n \times n$ matrices itself makes up something called a non-commutative Ring, not a topic for this class but worth a google if you are interested. A Ring, in short, is an algebraic structure for which its members satisfy specific properties.

So we know how to add vectors but how we add matrices? If $A, B \in R^{n \times m}$, the set of $n \times m$ matrices.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} \quad + \quad B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1m} \\ b_{21} & b_{22} & \cdots & b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & b_{n2} & \cdots & b_{nm} \end{bmatrix}$$

Then

$$A + B = \begin{bmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \cdots & a_{1m} + b_{1m} \\ a_{21} + b_{21} & a_{22} + b_{22} & \cdots & a_{2m} + b_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} + b_{n1} & a_{n2} + b_{n2} & \cdots & a_{nm} + b_{nm} \end{bmatrix}$$

That is we add them component wise.

Example: Calculate

$$A + B = \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 2 & 7 & 4 \\ 0 & 8 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 4 & 3 \\ 6 & 1 & 0 \\ 2 & 5 & 2 \\ 9 & 1 & 10 \end{bmatrix}$$

$$= \begin{bmatrix} 1+2 & 2+4 & 3+3 \\ -1+6 & 0+1 & 2+0 \\ 2+2 & 7+5 & 4+2 \\ 0+9 & 8+1 & 1+10 \end{bmatrix} =$$

What about $B + A$?

$$\begin{bmatrix} 2+1 & 4+2 & 3+3 \\ 6+(-1) & 1+0 & 0+2 \\ 2+2 & 5+7 & 2+4 \\ 9+0 & 1+8 & 10+1 \end{bmatrix}$$

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Scalar Multiplication

Let $c \in R$ and $A \in R^{n \times m}$ be a scalar then

$$\underline{cA} = c\underline{A} = c \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nm} \end{bmatrix} = \begin{bmatrix} \underline{ca_{11}} & \underline{ca_{12}} & \cdots & \underline{ca_{1m}} \\ \underline{ca_{21}} & \underline{ca_{22}} & \cdots & \underline{ca_{2m}} \\ \vdots & \vdots & \ddots & \vdots \\ \underline{ca_{n1}} & \underline{ca_{n2}} & \cdots & \underline{ca_{nm}} \end{bmatrix}$$

Example: Calculate

$$4 \begin{bmatrix} 1 & 2 & 3 \\ -1 & 0 & 2 \\ 2 & 7 & 4 \\ 0 & 8 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 4(1) & 4(2) & 4(3) \\ 4(-1) & 4(0) & 4(2) \\ 4(2) & 4(7) & 4(4) \\ 4(0) & 4(8) & 4(1) \end{bmatrix}$$

Theorem 3.11: Let s and t be scalars and $\underline{A}, \underline{B}, \underline{C} \in R^{n \times m}$ and $\underline{0_{nm}}$ a matrix of zeros of size $n \times m$ then,

- (a) $A + B = B + A$
- (b) $s(A + B) = sA + sB$
- (c) $(s + t)A = sA + tA$
- (d) $(A + B) + C = A + (B + C)$
- (e) $(st)A = s(tA)$
- (f) $(A + 0_{nm}) = A$

$$\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0_{2,3}$$

0_{nm} called additive Identity

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$$(n \times k)(k \times m) = n \times m$$

Matrix Multiplication

Let $A = [\vec{a}_1 \ \vec{a}_2 \ \dots \ \vec{a}_k] \in R^{n \times k}$ where $\vec{a}_i \in R^n$ and $B = [\vec{b}_1 \ \vec{b}_2 \ \dots \ \vec{b}_m] \in R^{k \times m}$ and $\vec{b}_i \in R^k$ then the matrix product is defined by:

$$\begin{aligned} AB &= [A\vec{b}_1 \ A\vec{b}_2 \ \dots \ A\vec{b}_m] = C \\ &= [(b_{11}\vec{a}_1 + b_{21}\vec{a}_2 + \dots + b_{k1}\vec{a}_k) \ (b_{12}\vec{a}_1 + b_{22}\vec{a}_2 + \dots + b_{k2}\vec{a}_k) + \dots + (b_{1m}\vec{a}_1 + b_{2m}\vec{a}_2 + \dots + b_{km}\vec{a}_k)] \\ &= \left(\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} b_{11} + \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} b_{21} + \dots + \begin{bmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{nk} \end{bmatrix} b_{k1} \right) + \left(\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} b_{12} + \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} b_{22} + \dots + \begin{bmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{nk} \end{bmatrix} b_{k2} \right) + \dots + \left(\begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix} b_{1m} + \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{n2} \end{bmatrix} b_{2m} + \dots + \begin{bmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{nk} \end{bmatrix} b_{km} \right) \\ &= \begin{bmatrix} b_{11}a_{11} + b_{21}a_{12} + \dots + b_{k1}a_{1k} & b_{12}a_{11} + b_{22}a_{12} + \dots + b_{k2}a_{1k} & \dots & b_{1m}a_{11} + b_{2m}a_{12} + \dots + b_{km}a_{1k} \\ b_{11}a_{21} + b_{21}a_{22} + \dots + b_{k1}a_{2k} & b_{12}a_{21} + b_{22}a_{22} + \dots + b_{k2}a_{2k} & \dots & b_{1m}a_{21} + b_{2m}a_{22} + \dots + b_{km}a_{2k} \\ \vdots & \vdots & \ddots & \vdots \\ b_{11}a_{n1} + b_{21}a_{n2} + \dots + b_{k1}a_{nk} & b_{12}a_{n1} + b_{22}a_{n2} + \dots + b_{k2}a_{nk} & \dots & b_{1m}a_{n1} + b_{2m}a_{n2} + \dots + b_{km}a_{nk} \end{bmatrix} \end{aligned}$$

We can also state this in the following way if $AB = C$ then

Which is an $n \times m$ matrix $c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \dots + a_{ik}b_{kj}$ $\leftarrow AB = C \begin{bmatrix} c_{11} & c_{12} & \dots \\ c_{21} & \dots & \dots \end{bmatrix}$

Example: Calculate

$$AB = \begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ 4 & 2 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ -1 & 2 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & 1 & 1 \\ 4 & 2 & -1 & 0 \end{bmatrix} =$$

$$\begin{bmatrix} (1)(2) + (0)(4) & (1)(-2) + (0)(2) & (1)(1) + (0)(-1) & (1)(1) + (0)(0) \\ (-1)(2) + (2)(4) & (-1)(-2) + (2)(2) & (-1)(1) + (2)(-1) & (-1)(1) + (2)(0) \\ (3)(2) + (1)(4) & (3)(-2) + (1)(2) & (3)(1) + (1)(-1) & (3)(1) + (1)(0) \end{bmatrix}$$

One way to think about it is that the entry in the i^{th} row and j^{th} column will come from the i^{th} row of the left matrix and the j^{th} column of the right matrix.

What about BA ?

$$(2 \times 3)(2 \times 4)$$

not same can't multiply!

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In General $AB \neq BA$ for matrices!

Matrices of different sizes

```
octave:2> A=[1,2,3,4;5,6,7,8]
```

A =

$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \quad \begin{matrix} 2 \times 4 \\ (2 \times 4)(4 \times 2) \end{matrix}$$

```
octave:3> B=[1 2; 3 4; 5 6; 7 8]
```

B =

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \\ 7 & 8 \end{bmatrix} \quad \begin{matrix} 4 \times 2 \\ (4 \times 2)(2 \times 4) \end{matrix}$$

```
octave:4> A*B
```

ans =

$$\begin{bmatrix} 50 & 60 \\ 114 & 140 \end{bmatrix}$$

```
octave:5> B*A
```

ans =

$$\begin{bmatrix} 11 & 14 & 17 & 20 \\ 23 & 30 & 37 & 44 \\ 35 & 46 & 57 & 68 \\ 47 & 62 & 77 & 92 \end{bmatrix}$$

Matrices of the same size

```
octave:6> A=[1,2,3;4,5,6;7,8,9]
```

A =

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

```
octave:7> B=[9,8,7;6,5,4;3,2,1]
```

B =

$$\begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

```
octave:8> A*B
```

ans =

$$\begin{bmatrix} 30 & 24 & 18 \\ 84 & 69 & 54 \\ 138 & 114 & 90 \end{bmatrix}$$

```
octave:9> B*A
```

ans =

$$\begin{bmatrix} 90 & 114 & 138 \\ 54 & 69 & 84 \\ 18 & 24 & 30 \end{bmatrix}$$

```
octave:10> A.*B
```

ans =

$$\begin{bmatrix} 9 & 16 & 21 \\ 24 & 25 & 24 \\ 21 & 16 & 9 \end{bmatrix}$$

```
octave:11> B.*A
```

ans =

$$\begin{bmatrix} 9 & 16 & 21 \\ 24 & 25 & 24 \\ 21 & 16 & 9 \end{bmatrix}$$

Notice here they were the same, but this was **not** matrix multiplication. In Matlab or Octave * and .* (dot times) are very different. The .* operator multiplies component wise $a_{11}b_{11}$ $a_{12}b_{12}$ etc. In fact it is the Hadamard product we mentioned when first talking about vectors. If the matrices are of different sizes you cannot do the Hadamard product.

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We stated in Theorem 3.11 that $A + 0_{nm} = A$ the zero matrix is often referred to as the additive identity for matrix addition, one might wonder then, is there a multiplicative identity for matrix multiplication?

The identity matrix: Let $A \in R^{n \times m}$ then the identity matrices will be $I_m \in R^{m \times m}$, $I_n \in R^{n \times n}$ and will have ones on the diagonal and zeros everywhere else. That is:

$$I = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

$n \times m$ $m \times m$ $n \times n$ $n \times m$

Further $AI_m = I_n A = A$

If $A \in R^{n \times n}$, is a square matrix $n = m$, then $I = I_n = I_m \in R^{n \times n}$ and we can say $AI = IA = A$

Example:

Multiply:

2×2 2×2

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1+0(3) & 2(1)+0(4) \\ 0(1)+1(3) & 0(2)+1(4) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} (1)(1)+2(0) & (1)(0)+2(1) \\ 3(1)+4(0) & 3(0)+4(1) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

3×3 3×3

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

2×3

3×2 2×3

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 3 \times 2$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$\begin{bmatrix} (1)(1)+2(0) & (1)(0)+2(1) \\ (3)(1)+4(0) & 3(0)+4(1) \\ 5(1)+6(0) & 5(0)+6(1) \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

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Theorem 3.13: Let s be a scalar and A, B, C be matrices of appropriate sizes. Then

- (a) $\underline{A}(\underline{BC}) = (\underline{AB})\underline{C}$
- (b) $\underline{A}(\underline{B} + \underline{C}) = \underline{AB} + \underline{AC}$
- (c) $(\underline{A} + \underline{B})\underline{C} = \underline{AC} + \underline{BC}$
- (d) $\underline{s}(\underline{AB}) = (\underline{sA})\underline{B} = \underline{A}(\underline{sB})$
- (e) $\underline{AI} = \underline{A}$
- (f) $\underline{IA} = \underline{A}$

Theorem 3.14 Let A, B, C be nonzero matrices then,

- (a) In general $\underline{AB} \neq \underline{BA}$
- (b) $\underline{AB} = 0$ **does not** imply that $\underline{A} = 0$ or $\underline{B} = 0$.
- (c) $\underline{AC} = \underline{BC}$ **does not** imply that $\underline{A} = \underline{B}$ or $\underline{C} = 0$

Example $\begin{matrix} 2 \times 2 & 2 \times 2 \\ \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} & \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \end{matrix}$

$$\begin{bmatrix} \boxed{-4 + 4} & \boxed{(1)(2) + -2} \\ \boxed{6 + 12} & \boxed{6 - 6} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Example Let $A = \begin{bmatrix} -3 & 3 \\ 11 & -3 \end{bmatrix}$, $B = \begin{bmatrix} -1 & 2 \\ 3 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}$ Find AB and BC

$$AC = \begin{bmatrix} 3 & 9 \\ 5 & 15 \end{bmatrix}$$

$$BC = \begin{bmatrix} 3 & 9 \\ 5 & 15 \end{bmatrix}$$

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The Transpose of a Matrix

The transpose of a matrix is easy, columns become rows and rows become rows become columns

The first column becomes the first row, the second column becomes the second row, the third column becomes the third row and so on.

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} \xrightarrow{\text{Transpose...}} A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}$$

Notice the diagonal stayed the same! Infact we are just moving a_{ij} to a_{ji} so on the diagonal $a_{ii} = a_{ii}$

Theorem 3.15: Let $A, B \in R^{n \times m}$, $C \in R^{m \times k}$ and s scalar. Then

- (a) $(A + B)^T = A^T + B^T$
- (b) $(sA)^T = sA^T$
- (c) $(AC)^T = C^T A^T$

Short Proof of (a) good enough for quiz credit as well

Let $A+B=C$ with $A=[a_{ij}]$, $B=[b_{ij}]$, $C=[c_{ij}]=[a_{ij}+b_{ij}]$
 Then let $A^T=[a_{ji}^T]$, $B^T=[b_{ji}^T]$, $C^T=[c_{ji}^T]$, by the definition
 of transposition, $a_{ji}^T=a_{ij}$, $b_{ji}^T=b_{ij}$, $c_{ji}^T=c_{ij}=a_{ij}+b_{ij}$.
 Since $A^T+B^T=[a_{ji}^T+b_{ji}^T]=[a_{ij}+b_{ij}]=C_{ji}=C^T_{ji}$
 and $C^T_{ji}=[a_{ij}+b_{ij}]^T=(A+B)^T \Rightarrow A^T+B^T=(A+B)^T$

Quiz problem: Prove (a)

We prove this by (1) calculating $A^T + B^T$ and (2) calculating $(A + B)^T$ and showing we get the same thing! Note tht we are fixing the values of the components in A and B and just moving them, so after the transposition the subscripts no longer correspond to the position in the matrix, rather the value itself. So if a_{32} was in row 3 col 2 in A and a_{23}^T in row 2 col 3 in A^T then $a_{23}^T = a_{32}$ which we put in the spot of a_{23}^T . I think the colors make things clear.

①

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}, B = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{bmatrix} \xrightarrow{\text{add}} A+B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & a_{mn}+b_{mn} \end{bmatrix}$$

↓ columns to rows ↓ columns to rows

$$A^T = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix}, B^T = \begin{bmatrix} b_{11} & b_{21} & \dots & b_{m1} \\ b_{12} & b_{22} & \dots & b_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ b_{1n} & b_{2n} & \dots & b_{mn} \end{bmatrix} \xrightarrow{\text{add}} A^T+B^T = \begin{bmatrix} a_{11}+b_{11} & a_{21}+b_{21} & \dots & a_{m1}+b_{m1} \\ a_{12}+b_{12} & a_{22}+b_{22} & \dots & a_{m2}+b_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n}+b_{1n} & a_{2n}+b_{2n} & \dots & a_{mn}+b_{mn} \end{bmatrix}$$

--- Same! ---

②

$$A+B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \dots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \dots & a_{2n}+b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \dots & a_{mn}+b_{mn} \end{bmatrix} \xrightarrow{\text{columns to rows}} (A+B)^T = \begin{bmatrix} a_{11}+b_{11} & a_{21}+b_{21} & \dots & a_{m1}+b_{m1} \\ a_{12}+b_{12} & a_{22}+b_{22} & \dots & a_{m2}+b_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n}+b_{1n} & a_{2n}+b_{2n} & \dots & a_{mn}+b_{mn} \end{bmatrix}$$

Section 3.2 Matrix Algebra

Powers of a matrix

Just like we would say x multiplied by itself n times is written x^n we can do the same with a matrix

$$A^n = \underbrace{AA \cdots A}_{n \text{ times}} \text{ where } A \text{ is multiplied by itself } n \text{ times}$$

Diagonal matrices are particularly nice for this operation.

$$A^n = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix} \cdots \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix} = \begin{bmatrix} a_{11}^n & 0 & 0 & \cdots & 0 \\ 0 & a_{22}^n & 0 & \cdots & 0 \\ 0 & 0 & a_{33}^n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn}^n \end{bmatrix}$$

Ex: Calculate A^3 if $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ $= \begin{bmatrix} 1^3 & 0 & 0 \\ 0 & 2^3 & 0 \\ 0 & 0 & 3^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 8 & 0 \\ 0 & 0 & 27 \end{bmatrix}$

Another important form of matrices are Upper Triangular and Lower Triangular matrices.

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

Upper Triangular

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

Lower Triangular

Note that an Upper triangular matrix corresponds to echelon form and a diagonal matrix is almost in reduced row echelon form. We can also make the following statements.

- (a) Upper and Lower Triangular matrices are called triangular
- (b) A Diagonal matrix is both Upper Triangular and Lower Triangular
- (c) If A and B are both either Upper Triangular or Lower Triangular so is $A + B$
- (d) Powers of triangular matrices are also triangular

Section 3.2 Matrix Algebra

Elementary Matrices

We have been doing row operations on matrices of late, it turns out that we can represent those row operations with matrix multiplication using Elementary Matrices.

We form an elementary matrix by doing a **single** desired row operation on the Identity matrix. After that row operation is done we can use the resulting matrix, call it E , and multiply it by a matrix A . The result of the multiplication will be the same as doing the row operation on A .

Example: Let $A = \begin{bmatrix} -2 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix}$ find a matrix E_1 such that $E_1 A$ produces the same result as swapping R_1, R_2 and a matrix E_2 such that $E_2 A$ produces the result of multiplying Row 1 by 2 and adding it to Row 2, that is $2R_1 + R_2$.

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{R_1, R_2} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = E_1$$

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \xrightarrow{2R_1 + R_2} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = E_2$$

$$\begin{array}{r} 2 \ 0 \\ + 0 \ 1 \\ \hline 2 \ 1 \end{array}$$

$$E_1 A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -2 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ -2 & 2 & -1 \end{bmatrix}$$

$$E_2 A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} -2 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 2 & -1 \\ -3 & 4 & -1 \end{bmatrix}$$

$$\begin{array}{r} 2R_1 + R_2 \quad \begin{array}{r} -4 \ 4 \ -2 \\ + 1 \ 0 \ 1 \\ \hline -3 \ 4 \ -1 \end{array} \end{array}$$

What is the effective result of $E_2 E_1 A$?

$$E_2(E_1 A) = E_2 \begin{bmatrix} 1 & 0 & 1 \\ -2 & 2 & -1 \end{bmatrix} \xrightarrow{2R_1 + R_2}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 3 \end{bmatrix}$$

Effective result

Section 3.2 Matrix Algebra

Composition of Linear Transformations.

In the last section we discussed how linear transformations on vectors in R^n can be represented by matrix multiplication. That is for any linear transformation $T_1(\vec{u})$ there is some matrix A_1 such that

$$T_1(\vec{u}) = A_1 \vec{u}$$

Imagine now that we have more than one linear transformation which we wish to compose

Let $T_1: R^n \rightarrow R^m$ and $T_2: R^m \rightarrow R^k$ The composition of linear transformations can be written

$$W(\vec{u}) = T_2(T_1(\vec{u})): R^n \rightarrow R^k$$

Suppose $T_1(\vec{u}) = A_1 \vec{u}$ and $T_2(\vec{v}) = A_2 \vec{v}$ Then $W(\vec{u}) = B \vec{u}$ Where $B = A_2 A_1$ and $B \in R^{k \times n}$

Example

Let $T_1 \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} x_1 + x_1 \\ x_1 - x_3 \\ 2x_1 + x_2 \end{bmatrix}$ $T_2 \left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \right) = \begin{bmatrix} x_1 + x_2 - x_3 \\ 2x_1 - x_2 - x_3 \end{bmatrix}$ Find A_1 and A_2

$$\begin{bmatrix} 2x_1 \\ x_1 - x_3 \\ 2x_1 + x_2 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

A_1

$$\begin{bmatrix} x_1 + x_2 - x_3 \\ 2x_1 - x_2 - x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

A_2

Find $A_2 A_1$

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & -1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 1 & 0 & -1 \\ 2 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$$

B

Section 3.2 Matrix Algebra

$$T_1 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 \\ x_1 - x_3 \\ 2x_1 + x_2 \end{pmatrix} \quad T_2 \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} x_1 + x_2 - x_3 \\ 2x_1 - x_2 - x_3 \end{pmatrix}$$

Calculate $T_2 \left(T_1 \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \right)$

$$T_2 \left(\begin{bmatrix} 2(1) \\ 1-1 \\ 2+0 \end{bmatrix} \right) = T_2 \left(\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix} \right) = \begin{bmatrix} 2+0-2 \\ 2(2)-0-2 \end{bmatrix}$$

$$T_2(T_1(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix})) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Verify $A_2 A_1 \vec{x}$ gives the same result for $\vec{x} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

$$\begin{matrix} 2 \times 3 & 3 \times 1 & 2 \times 1 \end{matrix}$$

$$\begin{bmatrix} 1 & -1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} (1)(1) + (-1)(0) + (-1)(1) \\ (1)(1) + (-1)(0) + (1)(1) \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 2 \end{bmatrix} \quad \checkmark$$

Section 3.2 Matrix Algebra

Partitioned Matrix

When we think about the size of a matrix $n \times m$ we can calculate how many numbers are in the matrix. nm . In many applied settings matrices can have tens of thousands or more rows and columns. So if a matrix has 10,000 rows and 10,000 columns you have 10^6 numbers. Then we want to add multiply and divide those numbers in computation, this can really eat up computer memory and CPU capacity. To aid in this we can split up our matrices into smaller matrices and carry out whatever computations a piece at a time to save RAM if needed or to parallelize the problem by working on several blocks at once.

$$A = \left[\begin{array}{cc|cc} 1 & -2 & 3 & 0 & 1 \\ 2 & 0 & 1 & 2 & 3 \\ -1 & 4 & 5 & 4 & 1 \\ 4 & 1 & 2 & 2 & 3 \end{array} \right]$$

A_{11} A_{12} A_{21} A_{22}

$$B = \left[\begin{array}{cc|cc} 0 & 1 & 4 & 0 & 2 \\ 2 & 1 & 2 & 4 & 4 \\ -1 & 3 & 5 & 6 & -1 \\ -2 & 3 & 1 & 0 & 2 \end{array} \right]$$

Here we have partitioned the matrices into what we call Block Matrices

$$A_{11} = \begin{bmatrix} 1 & -2 \\ 2 & 0 \\ -1 & 4 \end{bmatrix} \quad A_{12} = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 2 & 3 \\ 5 & 4 & 1 \end{bmatrix}$$

$$B_{11} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \\ -1 & 3 \end{bmatrix}$$

$$B_{12} = \begin{bmatrix} 4 & 0 & 2 \\ 2 & 4 & 4 \\ 5 & 6 & 1 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} 4 & 1 \end{bmatrix}$$

$$A_{22} = \begin{bmatrix} 2 & 2 & -3 \end{bmatrix}$$

$$B_{21} = \begin{bmatrix} -2 & 3 \end{bmatrix}$$

$$B_{22} = \begin{bmatrix} 1 & 0 & 2 \end{bmatrix}$$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

We can then represent our original matrices with the block matrices.

$$B = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

Addition

$$A + B = \begin{bmatrix} A_{11} + B_{11} & A_{12} + B_{12} \\ A_{21} + B_{21} & A_{22} + B_{22} \end{bmatrix}$$

Section 3.2 Matrix Algebra

Multiplying Partitioned Matrices

When adding partitioned matrices the matrices needed to be the same size and partitioned in the same way. However, when we multiply matrices they don't necessarily need to be the same size. In addition we have to make sure we partition them in a way that the block matrices themselves can be multiplied. That is so that their inner dimensions agree.

Example.

$$A = \left[\begin{array}{cc|cc|c} 1 & -2 & 3 & 0 & 1 \\ 2 & 0 & 1 & 2 & 3 \\ \hline -1 & 4 & 5 & 4 & 1 \\ 4 & 1 & 2 & 2 & -3 \end{array} \right]$$

$$B = \left[\begin{array}{c|c|c} 1 & 0 & 4 \\ -1 & 1 & 2 \\ \hline 5 & 2 & 2 \\ -2 & 1 & 0 \\ \hline 1 & 1 & 2 \end{array} \right]$$

$$A_{11} = \begin{bmatrix} 1 & -2 \\ 2 & 0 \\ -1 & 4 \end{bmatrix} \quad A_{12} = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 2 & 3 \\ 5 & 4 & 1 \end{bmatrix}$$

$$B_{11} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad B_{12} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad B_{13} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

$$A_{21} = \begin{bmatrix} 4 & 1 \end{bmatrix} \quad A_{22} = \begin{bmatrix} 2 & 2 & -3 \end{bmatrix}$$

$$B_{21} = \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix} \quad B_{22} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \quad B_{23} = \begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}$$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \quad \text{↖}$$

$$B = \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \end{bmatrix}$$

$$AB = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} & B_{13} \\ B_{21} & B_{22} & B_{23} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} & A_{11}B_{13} + A_{12}B_{23} \\ A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} & A_{21}B_{13} + A_{22}B_{23} \end{bmatrix}$$

Section 3.2 Matrix Algebra

$$\begin{matrix} 3 \times 2 \\ \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \end{matrix}$$

$$\begin{matrix} 2 \times 3 \\ \rightarrow \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix} \end{matrix}$$