

Section 3.4 LU factorization

Triangular matrices are nice, upper triangular matrices provide for easy back substitution and all those zeros in the upper or lower corners can make matrix multiplication much easier. Nowadays we typically use computers for all of this, but matrices can be enormous for application. The google page rank algorithm can have matrices with billions of entries! Further, in many applications we design a computer algorithm that converges to a solution after many iterations, the convergence of those algorithms can depend on matrix structure. One such structure is the triangular matrix.

Theorem 3.28

Let A be an $n \times m$ matrix. If A can be transformed to echelon form without interchanging rows, then there exists an $n \times n$ unit (ones on the diagonal) lower triangular matrix L and $n \times m$ echelon matrix U such that $A = LU$. If A is square, then U is also upper triangular.

The key here is that we won't interchange rows. If you recall we can find the inverse matrix of an invertible matrix A by multiplying all of the elementary matrices which result from doing the same row operations on the identity matrix I that we did to A to get A into echelon form. Let's think about the possible form for the product of elementary matrices **for which no rows were exchanged or multiplied by a number alone**.

$$I = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

It is also important to note, here we are only going to echelon form, that means we will not be worrying about introducing zeros above pivot positions, **only below them**.

In this special case, because we don't interchange rows and stop at echelon form in the end the product of elementary matrices will be lower triangular with ones on the diagonal!

$$E_n \cdots E_2 E_1 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ \# & 1 & 0 & \cdots & 0 \\ \# & \# & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \# & \# & \# & \cdots & 1 \end{bmatrix}$$

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So we have this nice equation

$$E_n \cdots E_2 E_1 A = U$$

Where we know U is in echelon form because that is what the product of the elementary matrices does to A , by design.

So if we multiply by the inverse of the elementary matrix product $(E_n \cdots E_2 E_1)^{-1}$ We have the equation

$$(E_n \cdots E_2 E_1)^{-1} (E_n \cdots E_2 E_1) A = (E_n \cdots E_2 E_1)^{-1} U$$

$$A = (E_n \cdots E_2 E_1)^{-1} U$$

And since the inverse of a lower triangular matrix is also lower triangular both $(E_n \cdots E_2 E_1)^{-1}$ and $E_n \cdots E_2 E_1$ are lower triangular. So we just define

$$L = (E_n \cdots E_2 E_1)^{-1}$$

And we have that

$$A = LU$$

With L lower triangular and U in echelon form!

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So what? Well other than it is cool we can factor a matrix this way these form make their way into computer science, numerical solutions of PDE's , optimization algorithms, statistics . All because they can reduce computational cost once known. In many cases you end up using the same matrix over and over in application, computation can be sped up or special algorithms used if you have the LU factorization.

Example: Find the LU factorization for the matrix A

$$A = \begin{bmatrix} 2 & -2 \\ 2 & 1 \\ 0 & -9 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$-R_1 + R_2$$

$$3R_2 + R_3$$

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So we know that $L^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 3 & 1 \end{bmatrix}$ Lets find L

Let's verify that $L = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$ times $U = \begin{bmatrix} 2 & -3 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}$ equals $A = \begin{bmatrix} 2 & -2 \\ 2 & 1 \\ 0 & -9 \end{bmatrix}$

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Use the given LU factorization to find all solutions to $A\mathbf{x} = \mathbf{b}$.

$$A = \begin{bmatrix} 1 & 0 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 5 \\ 0 & -3 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -9 \\ -30 \end{bmatrix}$$

2a

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```
octave:2> A=[1,2,3,4;5 6 7 8; 9 10 11 12; 13 14 15 16]
A =
```

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16

```
octave:3> format rat
```

```
octave:4> [L,U]=lu(A)
```

```
L =
```

1/13	1	0	0
5/13	2/3	-1/2	1
9/13	1/3	1	0
1	0	0	0

```
U =
```

13	14	15	16
0	12/13	24/13	36/13
0	0	0	0
0	0	0	0

```
octave:5> A=[2 -2; 2 1; 0 -9]
```

```
A =
```

2	-2
2	1
0	-9

```
octave:6> [L,U]=lu(A)
```

```
L =
```

1	0
1	-1/3
0	1

```
U =
```

2	-2
0	-9