

Section 4.1 Into to Subspaces

A subspace is a subset of a set of vectors which has properties that make it act like a whole vector space. Namely, if we add two vectors we don't leave the subspace and if we scale a vector also don't leave the subspace. This is nice, imagine if we could add two vectors in R^2 and end up in R^3 that would be like being able to walk two steps left and one step right but end up 300 ft in the air.

A subset of R^n is a **subspace** if S satisfies the following three conditions

- (a) S contains $\vec{0}$
- (b) if $\vec{u}$ and $\vec{v}$ are in S, the $\vec{u} + \vec{v}$ is also in S. (closure under addition)
- (c) If r is a real number and $\vec{u}$ is in S, then $r\vec{u}$ is also in S. (closure under scalar multiplication)

Example: Show that $S = \{\vec{u} \in R^3 \mid u_2 = 0\}$ is a subspace of R^3 .

Such that $u_2 = 0$

$$\begin{bmatrix} a \\ 0 \\ b \end{bmatrix} \in S$$

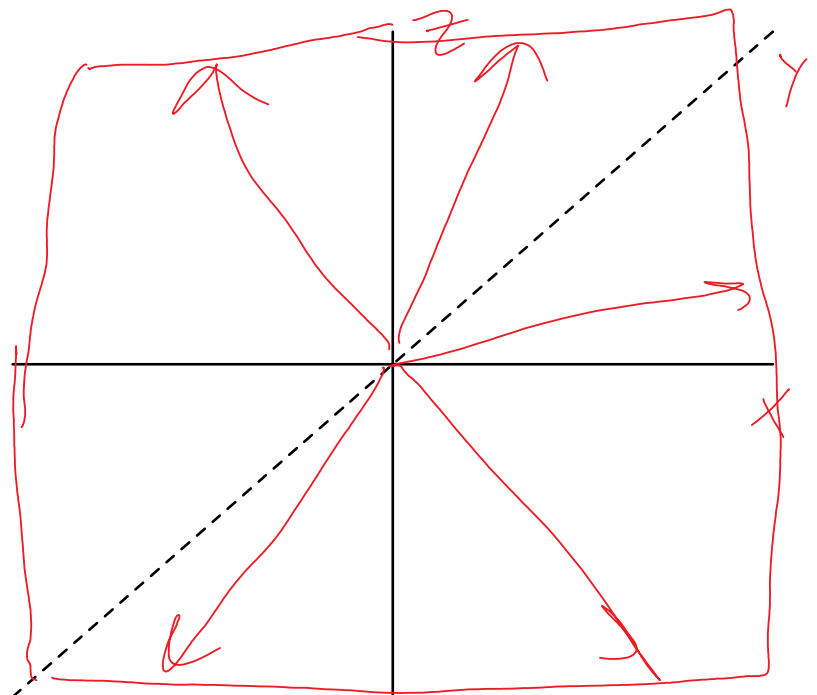
Is

$$a) \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in S? \quad u_2 = 0$$

$$b) \vec{u} = \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} \quad \vec{v} = \begin{bmatrix} a_1 \\ 0 \\ b_1 \end{bmatrix}, \text{ Is } \vec{u} + \vec{v} \in S?$$

$$\begin{bmatrix} a \\ 0 \\ b \end{bmatrix} + \begin{bmatrix} a_1 \\ 0 \\ b_1 \end{bmatrix} = \begin{bmatrix} a+a_1 \\ 0+0 \\ b+b_1 \end{bmatrix} = \begin{bmatrix} a+a_1 \\ 0 \\ b+b_1 \end{bmatrix} \quad \checkmark$$

$$c) r \begin{bmatrix} a \\ 0 \\ b \end{bmatrix} = \begin{bmatrix} ra \\ r0 \\ rb \end{bmatrix} = \begin{bmatrix} ra \\ 0 \\ rb \end{bmatrix}$$



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Rapid fire, Why **aren't** these subspaces?

A subset of R^n is a **subspace** if S satisfies the following three conditions

(a) S contains $\vec{0}$

(b) if $\vec{u}$ and $\vec{v}$ are in S, the $\vec{u} + \vec{v}$ is also in S.

(c) If r is a real number and U is in S, then $r\vec{u}$ is also in S.
(closure under scalar multiplication)

- 1) The set of vectors in R^2 such that the second component is not zero $S = \{\vec{u} \in R^2 \mid \vec{u} = \begin{bmatrix} a \\ b \end{bmatrix}, b \neq 0\}$
 such that

$\begin{bmatrix} 0 \\ b \end{bmatrix} \quad b \neq 0 \quad \vec{0} \notin S$ Not a subspace

- 2) The set of vectors in R^4 where at least one of the components of the vector is zero.

$S = \{\vec{u} \in R^4 \mid \exists u_i = 0 \text{ for at least one } i \in \{1,2,3,4\}\}$
 s.t. There exists

$\begin{bmatrix} 1 \\ 2 \\ 3 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 6 \\ 0 \\ 4 \end{bmatrix}$
 $\vec{u} \quad \vec{v}$

a) $\vec{0} \in S$

b) $\vec{u} + \vec{v} = \begin{bmatrix} 1+1 \\ 2+0 \\ 3+3 \\ 0+2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 6 \\ 2 \end{bmatrix} \notin S$ Not a subspace

- 3) The set of vectors in R^3 with length equal to 1 or 0, $S = \{\vec{u} \in R^3 \mid |\vec{u}| = 1 \text{ or } |\vec{u}| = 0\}$

c) $r\vec{u} \in S \quad |r\vec{u}| = r|\vec{u}| = r(1) = r$

$2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \left| \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right| = \sqrt{2^2 + 0^2} = \sqrt{4} = 2 \neq 1$
 $\downarrow$
 $\sqrt{1^2 + 0} = 1$

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Is it a subspace?

A subset of R^n is a **subspace** if S satisfies the following three conditions

$$S = \{\vec{u} \in R^3 | u_1 + u_2 + u_3 = 0\}$$

- (a) S contains $\vec{0}$
- (b) if $\vec{u}$ and $\vec{v}$ are in S , the $\vec{u} + \vec{v}$ is also in S .
- (c) If r is a real number and $\vec{v}$ is in S , then $r\vec{v}$ is also in S .

a) $I_S \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in S \quad \underline{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \Rightarrow \underline{v_1 + v_2 + v_3 = 0 + 0 + 0 = 0}$

$$\vec{o} \in S$$

b) let $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in S$
 $\downarrow$
 $v_1 + v_2 + v_3 = 0$

$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in S$
 $\downarrow$
 $v_1 + v_2 + v_3 = 0$

$$\vec{U} + \vec{V} = \begin{bmatrix} U_1 \\ U_2 \\ U_3 \end{bmatrix} + \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} U_1 + V_1 \\ U_2 + V_2 \\ U_3 + V_3 \end{bmatrix} \quad \text{If } \in S$$

the I need $(v_1 + v_1) + (v_2 + v_2) + (v_3 + v_3) \stackrel{?}{=} 0$

re arrange $(\underbrace{V_1 + V_2 + V_3}_0) + (\underbrace{V_1 + V_2 + V_3}_0) = 0 \checkmark$

c) let $\vec{v} \in S \Rightarrow \underline{v_1 + v_2 + v_3 = 0}$

$$\vec{r} \vec{v} = r \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} r v_1 \\ r v_2 \\ r v_3 \end{bmatrix} \text{ If in } S \text{ then } r v_1 + r v_2 + r v_3 = 0$$

$$\rightarrow r(u_1 + u_2 + u_3) = r(0) = 0 \checkmark$$

⇒ S is a subspace to $a), b), c)$

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Something that is always a subspace

A subset of R^n is a **subspace** if S satisfies the following three conditions

(a) S contains $\vec{0}$

(b) if $\vec{u}$ and $\vec{v}$ are in S , the $\vec{u} + \vec{v}$ is also in S .

(c) If r is a real number and U is in S , then $r\vec{u}$ is also in S .

If we define $S = \text{span}\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\}$ with $\vec{u}_i \in R^n$ Then S is a subspace of R^n .

Proof:

S is the set of all possible linear combos of $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\}$

$$\vec{v} = c_1\vec{u}_1 + c_2\vec{u}_2 + \dots + c_m\vec{u}_m$$

a) is $\vec{0} \in S$ take $c_1, c_2, \dots, c_m = 0$

$$0\vec{u}_1 + 0\vec{u}_2 + \dots + 0\vec{u}_m = \vec{0} \checkmark$$

b) $\vec{u}, \vec{v} \in S$ then is $\vec{u} + \vec{v} \in S$

assume $\vec{u}, \vec{v} \in S$

$$\Rightarrow \vec{u} = c_1\vec{u}_1 + c_2\vec{u}_2 + \dots + c_m\vec{u}_m$$

$$\vec{v} = c'_1\vec{u}_1 + c'_2\vec{u}_2 + \dots + c'_m\vec{u}_m$$

$$\vec{u} + \vec{v} = (c_1\vec{u}_1 + \dots + c_m\vec{u}_m) + (\underline{c'_1\vec{u}_1} + \dots + c'_m\vec{u}_m)$$

$$\vec{u} + \vec{v} = (c_1 + c'_1)\vec{u}_1 + (c_2 + c'_2)\vec{u}_2 + \dots + (c_m + c'_m)\vec{u}_m \checkmark$$

c) $\vec{u} \in S$ then is $r\vec{u}$ in S ?

$$\vec{v} = c_1 \vec{v}_1 + \dots + c_m \vec{v}_m \rightarrow r\vec{v} = r(c_1 \vec{v}_1 + \dots + c_m \vec{v}_m) = rc_1 \vec{v}_1 + \dots + rc_m \vec{v}_m$$

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Null Space

The null space of a linear transformation $A\vec{x}$ is the set of vectors which get mapped to the zero vector by A .

$$Null(A) = \{\vec{x} \mid A\vec{x} = \vec{0}\}$$

Theorem: The null space of a linear transformation $A\vec{x}$ where $A \in R^{n \times m}$ is a subspace of R^n

Proof:

a) Is $\vec{0} \in Null(A)$? If so $A\vec{0} = \vec{0}$ ✓

$$\vec{0} \in Null(A)$$

$$\begin{bmatrix} a_{11} & \dots \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = 0$$

b) If $\vec{u}, \vec{v} \in Null(A)$ is $\vec{u} + \vec{v} \in Null(A)$?

$$A\vec{u} = \vec{0}$$

$$A\vec{v} = \vec{0}$$

$$A(\vec{u} + \vec{v}) = \underbrace{A\vec{u}}_{\vec{0}} + \underbrace{A\vec{v}}_{\vec{0}} = \vec{0} \checkmark$$

c) If $\vec{u} \in Null(A)$ $r\vec{u} \in Null(A)$
 $\downarrow$?

$$A\vec{u} = \vec{0}$$

$$A(r\vec{u}) = r(A\vec{u}) = r(\vec{0}) = \vec{0} \checkmark$$

What happens when the only vector in the null space of A is the zero vector

Theorem 3.5: Let T be a linear transformation. Then T is one-to-one if and only if the only solution to $T(\vec{x}) = \vec{0}$ is the trivial solution $\vec{x} = \vec{0}$.

$$A\vec{x} = \vec{0}$$

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Find the null space for the given matrix. (A graphing calculator is recommended.)

$$A = \begin{bmatrix} -1 & 0 & -0 & -4 & 5 & 4 \\ 6 & 2 & 1 & 2 & 5 & 0 \\ 3 & 2 & -5 & -1 & 0 & 4 \end{bmatrix}$$

8a

3x6

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix}$$

$$\text{null}(A) = \text{span} \left\{ \begin{bmatrix} \boxed{} \\ \boxed{} \\ \boxed{} \\ \boxed{} \\ \boxed{} \\ \boxed{} \end{bmatrix} \right\}$$

$$[4, 5, 4; -47/4, -95/6, -34/3; -5/2, -10/3, -4/3; 1, 0, 0; 0, 1, 0; 0, 0, 1]$$

want to find all $\vec{x} \in \mathbb{R}^6$

$$A\vec{x} = \vec{0}$$

$$3 \times 6, 6 \times 1 = 3 \times 1$$

$$\begin{bmatrix} -1 & 0 & 0 & 4 & 5 & 4 \\ 6 & 2 & 1 & 2 & 5 & 0 \\ 3 & 2 & -5 & -1 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{cccccc|c} -1 & 0 & 0 & 4 & 5 & 4 & 0 \\ 6 & 2 & 1 & 2 & 5 & 0 & 0 \\ 3 & 2 & -5 & -1 & 0 & 4 & 0 \end{array} \right]$$

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octave:2> A=[-1 0 0 4 5 4 0; 6 2 1 2 5 0 0; 3 2 -5 -1 0 4 0]

A =

-1	0	0	4	5	4	0
6	2	1	2	5	0	0
3	2	-5	-1	0	4	0

octave:3> format rat

octave:4> rref(A)

ans =

x_1	x_2	x_3	x_4	x_5	x_6	
1	0	0	-4	-5	-4	0
0	1	0	47/4	95/6	34/3	0
0	0	1	5/2	10/3	4/3	0

$$x_6 = s_1, x_5 = s_2, x_4 = s_3$$

$$\begin{aligned} 0x_1 + 0x_2 + x_3 + 5/2 s_3 + 10/3 s_2 + 4/3 s_1 &= 0 \Rightarrow x_3 = -5/2 s_3 - 10/3 s_2 - 4/3 s_1 \\ 0x_1 + x_2 + 0x_3 + 47/4 s_3 + 95/6 s_2 + 34/3 s_1 &= 0 \Rightarrow x_2 = -47/4 s_3 - 95/6 s_2 - 34/3 s_1 \\ x_1 + 0x_2 + 0x_3 - 4s_3 - 5s_2 - 4s_1 &= 0 \Rightarrow x_1 = 4s_3 + 5s_2 + 4s_1 \end{aligned}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 4s_3 + 5s_2 + 4s_1 \\ -47/4 s_3 - 95/6 s_2 - 34/3 s_1 \\ -5/2 s_3 - 10/3 s_2 - 4/3 s_1 \\ s_3 \\ s_2 \\ s_1 \end{bmatrix} = s_1 \begin{bmatrix} 4 \\ -34/3 \\ -4/3 \\ 0 \\ 0 \\ 1 \end{bmatrix} + s_2 \begin{bmatrix} 5 \\ -95/6 \\ -10/3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + s_3 \begin{bmatrix} 4 \\ -47/4 \\ -5/2 \\ 1 \\ 0 \\ 0 \end{bmatrix} =$$

$$\text{Null}(A) = \text{Span} \left\{ \begin{bmatrix} 4 \\ -34/3 \\ -4/3 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -95/6 \\ -10/3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -47/4 \\ -5/2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

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Another word for Null space, Kernel

Let $T: R^n \rightarrow R^m$ then the set of vectors $\vec{x}$ which satisfy $T(\vec{x}) = 0$ is called the Kernel of T

$$\text{Kernel}(T) = \{\vec{x} \in R^n | T(\vec{x}) = 0\}$$

For our linear transformations thus far, they can all be represented by matrices and as such Kernel and null space are exactly the same thing here. The terms Kernel and Null Space are often used interchangeably, although Kernel is typically used in more general contexts.

If you recall the derivative operator was a linear operator

$$\frac{d}{dx}(f(x) + g(x)) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x), \quad \frac{d}{dx}cf(x) = c \frac{d}{dx}f(x)$$

For the linear operator $\frac{d}{dx}$ we could say that $\text{Ker}\left(\frac{d}{dx}\right) = \{f(x) | f(x) = c \text{ where } c \text{ is constant}\}$

Let $T: R^n \rightarrow R^m$ where $T = A\vec{x}$ we say that the range of T, or equivalently the range of A, is a subspace of the codomain.

Proof:

The range of A is the span of the column vectors which make it up $A = [\vec{a}_1 \ \vec{a}_2 \ \dots \ \vec{a}_3]$ and we proved,

If we define $S = \text{span}\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\}$ with $\vec{u}_i \in R^n$ Then S is a subspace of R^n .

Thus the range of A is the span of a set of vectors in the codomain and must therefore be a subspace!

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Theorem 3.5: Let T be a linear transformation. Then T is one-to-one if and only if the only solution to $T(\vec{x}) = 0$ is the trivial solution $\vec{x} = 0$.

→ This is equivalent to the following theorem

Theorem: Let $T: R^n \rightarrow R^m$, then T is one-to-one if and only if the $\ker\{T\} = 0$

The Unifying Theorem Expanded again!

Let $S = \{\vec{a}_1, \dots, \vec{a}_n\}$ be a set of n vectors in R^n , let $A = [\vec{a}_1 \ \vec{a}_2 \ \dots \ \vec{a}_n]$. Let $T: R^n \rightarrow R^n$ be given by $T(\vec{x}) = A\vec{x}$. Then the following are equivalent:

- (a) S spans R^n
- (b) S is linearly independent
- (c) $Ax = b$ has a unique solution for all $\vec{b} \in R^n$
- (d) T is onto
- (e) T is one-to-one
- (f) A is invertible
- (g) $\ker(T)=0$ or equivalently the Nullspace(A)=0

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Comment on applications of Subspaces (optional example for the curious)

We like subspaces because they act like vector spaces, if we add to elements of a subspace, we are still in that space, if we scale one, we are still in that space. Subspaces can represent specific physical, economic or other states. If you know what the subspace looks like you gain knowledge of all possible states satisfying the constraints which define the space.

Let's do a silly but fun example. Suppose you are an engineer and are tasked with making a composite material made of a plastic into which you embed particles of 3 different masses of 2,4 and 6. These particles make the plastic photoelectric as long as the center of mass of the composite lies exactly at the center (0,0). If we need 100 sets of each of the 3 particles in each plastic to get our photoelectric properties how can they be distributed so that the center of mass (com) remains at (0,0)?

Let's start with the case of just 3 of each particle. For 3 particles in a plane the x and y coordinates of the center of mass are given by:

$$x_{com} = \frac{1}{M}(m_1x_1 + m_2x_2 + m_3x_3) \quad y_{com} = \frac{1}{M}(m_1y_1 + m_2y_2 + m_3y_3)$$

Where $M = m_1 + m_2 + m_3$, (x_1, y_1) are where m_1 is and $(x_2, y_2), (x_3, y_3)$ the positions of m_2 and m_3 respectively. Or as a vector equation:

$$\begin{bmatrix} x_{com} \\ y_{com} \end{bmatrix} = \frac{1}{M} \begin{bmatrix} m_1x_1 + m_2x_2 + m_3x_3 \\ m_1y_1 + m_2y_2 + m_3y_3 \end{bmatrix}$$

For our numbers we have:

$$\begin{bmatrix} x_{com} \\ y_{com} \end{bmatrix} = \frac{1}{12} \begin{bmatrix} 2x_1 + 4x_2 + 6x_3 \\ 2y_1 + 4y_2 + 6y_3 \end{bmatrix}$$

What we want is for $x_{com} = 0, y_{com} = 0$ which reads as

$$\frac{1}{12} \begin{bmatrix} 2x_1 + 4x_2 + 6x_3 \\ 2y_1 + 4y_2 + 6y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

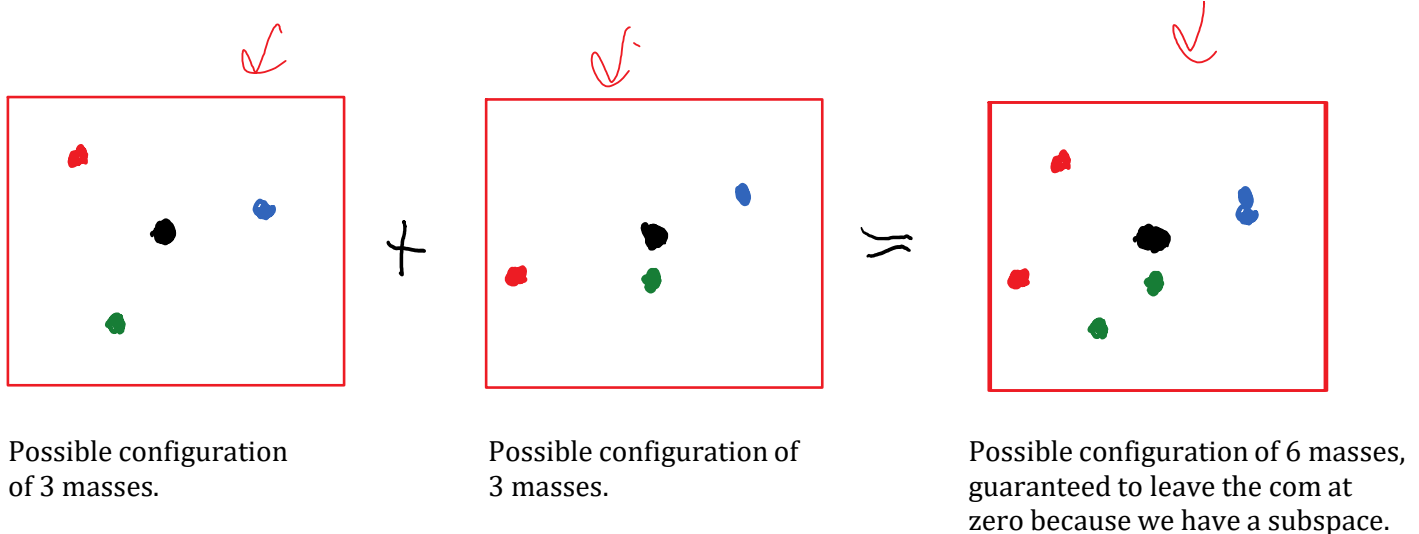
Multiply both sides by 12 and rewrite as a matrix equation and we get:

$$\begin{bmatrix} 2 & 4 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The solutions we seek can be put into a vector in R^6 and indeed we want the vectors which our matrix maps to 0! It turns out we want the **null space** of the matrix above! Further, the nullspace is a **subspace** of R^6 by our theorem.

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Why do we care that the solution to the 3 mass problem is a subspace? Well because the sum of any set of the vectors in the null space will also be in the null space. This means that if we find the vectors that span the null space we can simply take any linear combination of them to get a vector which represents x,y coordinates at which each mass could be placed to keep the com at zero. If we take two such vectors we can add them and the center of mass will still be at zero because their sum will be in the subspace. This means if we find two different sets of x,y coordinates for which the com of mass is zero, then we could place 6 masses at those points and we are guaranteed that the com is still zero.



So... we could take 100 linear combinations of our null space, each of which represents a set of configurations of the 3 masses where the com=0 lay put the masses at all 300 of those (x,y) points and the com will still be zero because subspaces are closed under addition! An easy to solve the problem without solving it for 300 masses total! As a bonus we can randomly take those linear combinations to find different configurations that work.

$$\begin{bmatrix} 2 & 4 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 4 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

If we examine the augmented system

$$\begin{bmatrix} 2 & 4 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 4 & 6 & 0 \end{bmatrix}$$

We see that we are already in echelon form! We also see that we have free variables for y_3, y_2, x_3, x_2

$$\text{So let } y_3 = s_1, \quad y_2 = s_2, \quad x_3 = s_3, \quad x_2 = s_4$$

Our linear system which we could back solve is then given by

$$\begin{aligned} 2x_1 + 4x_2 + 6x_3 &= 0 \\ x_2 &= s_4 \\ x_3 &= s_3 \\ 2y_1 + 4y_2 + 6y_3 &= 0 \\ y_2 &= s_2 \\ y_3 &= s_1 \end{aligned}$$

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If we back solve

$$\begin{aligned}
 2x_1 + 4x_2 + 6x_3 &= 0 \\
 x_2 &= s_4 \\
 x_3 &= s_3 \\
 2y_1 + 4y_2 + 6y_3 &= 0 \\
 y_2 &= s_2 \\
 y_3 &= s_1
 \end{aligned}$$

We obtain

$$\begin{aligned}
 x_1 &= -2s_4 - 3s_3 \\
 x_2 &= s_4 \\
 x_3 &= s_3 \\
 y_1 &= -2s_2 - 3s_1 \\
 y_2 &= s_2 \\
 y_3 &= s_1
 \end{aligned}$$

Or as a vector

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -2s_3 - 3s_4 \\ s_3 \\ s_4 \\ -2s_2 - 3s_1 \\ s_2 \\ s_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ -3s_1 \\ 0 \\ s_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2s_2 \\ s_2 \\ 0 \end{bmatrix} + \begin{bmatrix} -2s_3 \\ s_3 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -3s_4 \\ 0 \\ s_4 \\ 0 \\ 0 \\ 0 \end{bmatrix} = s_1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix} + s_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + s_3 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s_4 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Note that the s 's could be anything we like, so that means our solution is just a linear combination of the vectors above, all possible configurations of the 3 masses make up the span of the vectors.

$$\text{solution space} = \text{span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} = \text{nullspace} \left\{ \begin{bmatrix} 2 & 4 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 4 & 6 \end{bmatrix} \right\}$$

And because the null space is a sub space if we add two vectors from it we get another in it!

A subset of R^n is a **subspace** if S satisfies the following three conditions

- (a) S contains $\vec{0}$
- (b) if $\vec{u}$ and $\vec{v}$ are in S , the $\vec{u} + \vec{v}$ is also in S . (closure under addition)
- (c) If r is a real number and U is in S , then $r\vec{u}$ is also in S . (closure under scalar multiplication)

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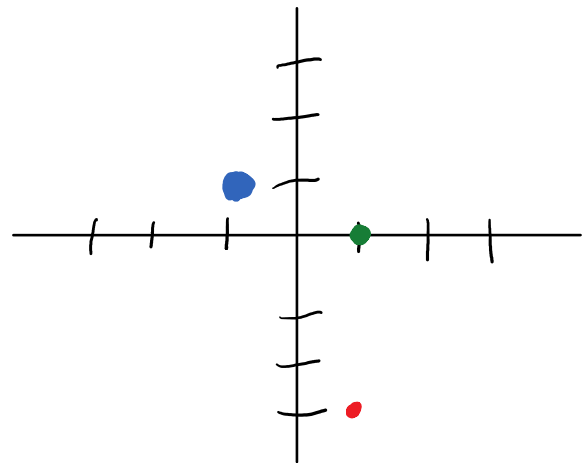
$$\text{solution space} = \text{span} \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ -3 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\} = \text{nullspace} \left\{ \begin{bmatrix} 2 & 4 & 6 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 4 & 6 \end{bmatrix} \right\}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = s_1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix} + s_2 \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + s_3 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s_4 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

A couple of solutions, choose $s_1 = 1, s_2 = 0, s_3 = 1, s_4 = -1$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = 1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} - 1 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -3 \\ 0 \\ 1 \end{bmatrix}$$

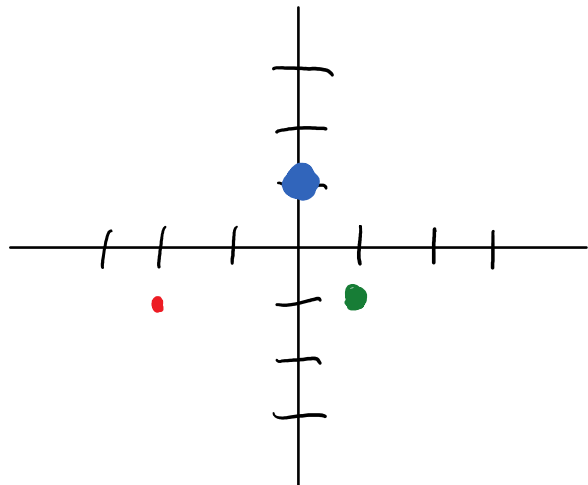
So mass 1 should be at (1,-3), mass 2 at (1,0), and mass 3 at (-1,1)



Choose $s_1 = 0, s_2 = -1, s_3 = 1, s_4 = 0$

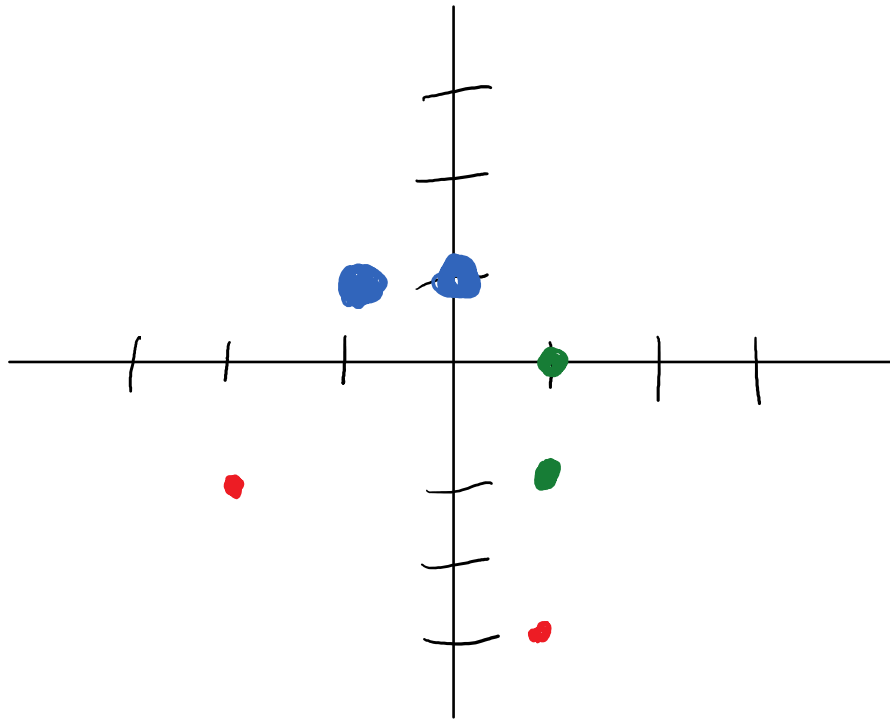
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} = 1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ -3 \\ 0 \\ 1 \end{bmatrix} - 1 \begin{bmatrix} 0 \\ 0 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + 0 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ -1 \\ -1 \\ 1 \end{bmatrix}$$

So mass 1 should be at (-2,-1), mass 2 at (1,-1), mass 3 at (0,1)



Section 4.1 Into to Subspaces

If we combine these two states then the com is still zero because we have a subspace!



Let's check! For 6 particles the com is given by

$$x_{com} = \frac{1}{M}(m_1x_1 + m_2x_2 + m_3x_3 + m_4x_4 + m_5x_5 + m_6x_6) \quad y_{com} = \frac{1}{M}(m_1y_1 + m_2y_2 + m_3y_3 + m_6y_4 + m_5y_5 + m_6y_6)$$

Here $M = 24$ and $m_1 = m_4 = 2$, $m_2 = m_5 = 4$, $m_3 = m_6 = 6$

$$x_{com} = \frac{1}{24}(2(1) + 4(1) + 6(-1) + 2(-2) + 4(1) + 6(0)) = 0$$

$$y_{com} = \frac{1}{24}(2(-3) + 4(0) + 6(1) + 2(-1) + 4(-1) + 6(1)) = 0$$

Success! We could keep going to get all 100 sets of 3 just by choosing random values s_1, s_2, s_3, s_4 each would represent positions of the 3 basic particles which give 0 com, and since our set is a subspace we could add different configurations to the mix and we are guaranteed that the com will still be at 0! No need to keep checking!

Section 4.1 Into to Subspaces

To motivate you to look at the octave code that goes along with this and to think about subspaces in a physical context, I am making the following offer. If you complete the questions below I will let you hand in the corresponding work on a quiz day of your choice instead of taking the quiz. You will simply receive full credit for the quiz should you choose to do this optional assignment. To get full credit for this quiz replacement you must do the following 3 things.

1. Choose 2 sets of values for s_1, s_2, s_3, s_4 and find the corresponding mass position vectors $\vec{u}_1, \vec{u}_2$ as on the previous page. You must choose different values that I did. Show by hand that the center of mass for each configuration and the combined configuration of all six particles is zero. Next add $\vec{u}_1$ and $\vec{u}_2$ to obtain a new position vector $\vec{u}_3$. This vector is also in our subspace so it's configuration will also have com 0, show this by hand as well. Sketch each set of configurations.

For the next parts you need to go to <https://octave-online.net/bucket~ERtupqxYUt8gy4ZoWsq7Lk>
This will give you access to some prewritten code that I want you to play with.

If you don't want to use octave online you can download the files for parts 2 and 3 from sakai in the recourses folder.

2. Use the com.m code you see in octave online to generate different sets of configurations of 3 particles that have center of mass 0. Pick two you like and print them out, you will hand it in with your answer to question 1.
3. Use the com100.m code to generate an image with 100 different sets of the three particles all in a configuration with center of mass zero. Pick one you like and print it out as you will hand it in with your work to questions 1 and 2.

The code is also on the next two pages. Anything with a % in front of is not code, but a comment.

Section 4.1 Into to Subspaces

%com.m

%each time you run this code you will be generating a different possible of configuration of the three masses which have the center of mass at zero.

% Lets set the vectors that span the null space

n1=[0;0;0;-3;0;1]

n2=[0;0;0;-2;1;0]

n3=[-2;1;0;0;0;0]

n4=[-3;0;1;0;0;0]

% now we will generate 4 random numbers in (-1,1) to be the coefficients of a linear combination of our 4 vectors

s1=-1+2*rand(1,1)

s2=-1+2*rand(1,1)

s3=-1+2*rand(1,1)

s4=-1+2*rand(1,1)

%next take a linear combination to get a possible configuration

config=s1*n1+s2*n2+s3*n3+s4*n4

% then since the structure of the configuration vector was [x1,x2,x3,y1,y2,y3] mass 1 at (x1,y1) mass 2 at (x2,y2) and mass 3 at (x3,y3) we pull these out of the config vector as

x1=config(1)

x2=config(2)

x3=config(3)

y1=config(4)

y2=config(5)

y3=config(6)

% lets calculate the center of mass

format rat % to avoid -0.0 stuff

xcom=(1/12)*(2*x1+4*x2+6*x3)

ycom=(1/12)*(2*y1+4*y2+6*y3)

format short % back to decimals

%lets plot them!

plot(x1,y1,'color','r','o')

hold on

plot(x2,y2,'color','g','o')

hold on

plot(x3,y3,'color','b','o')

hold on

plot(xcom,ycom,'*','color','m')

hold off

clear all

Section 4.1 Into to Subspaces

% With this code we will lay down 100 different configurations of the 3 masses and show that the com will still be zero! Note if you run this in octave online you will have to click the Resume Execution button by the command prompt a few times until the plot shows up. Ovtave online wants to make sure you are there if you give it anything hefty.

% again the vectors which span the null space

```
n1=[0;0;0;-3;0;1]
```

```
n2=[0;0;0;-2;1;0]
```

```
n3=[-2;1;0;0;0;0]
```

```
n4=[-3;0;1;0;0;0]
```

% this loop generates m configurations for the three masses

```
m=100;
```

```
for i=1:m
```

% now we will generate 4 random numbers in (-1,1) to be the coefficents of a linear combination of our 4 vectors in a loop m times, m solution vectors

```
s1=-1+2*rand(1,1)
```

```
s2=-1+2*rand(1,1)
```

```
s3=-1+2*rand(1,1)
```

```
s4=-1+2*rand(1,1)
```

%next take a linear combination to get a possible configuration

```
config(:,i)=s1*n1+s2*n2+s3*n3+s4*n4 % places the ith vector in the ith column of a matrix
```

```
end;% loop over
```

% then since the structure of the configuration vector was [x1,x2,x3,y1,y2,y3] mass 1 at (x1,y1) mass 2 at (x2,y2) and mass 3 at (x3,y3) we pull these out of the config vector as

%here I am grabbing the first row of the config matrix storing it represents all of the

%x coordinates for each solution

```
x1s=config(1,:)
```

```
x2s=config(2,:)
```

```
x3s=config(3,:)
```

```
y1s=config(4,:)
```

```
y2s=config(5,:)
```

```
y3s=config(6,:)
```

format rat % to avoid -0.0 stuff

% calculate the center of mass with all of the points.the sum function in this case adds all the components stored in the row vectors we multiply m by 12 since we have m sets of the 3 masses which total mass of 12

```
xcom=(1/(12*m))*(2*sum(x1s)+4*sum(x2s)+6*sum(x3s))
```

```
ycom=(1/(12*m))*(2*sum(y1s)+4*sum(y2s)+6*sum(y3s))
```

format short % back to decimals

%lets plot them!

```
plot(x1s,y1s,'color','r','o')
```

```
hold on
```

```
plot(x2s,y2s,'color','g','o')
```

```
hold on
```

```
plot(x3s,y3s,'color','b','o')
```

```
hold on
```

```
plot(xcom,ycom,'*','color','m')
```

```
hold off
```

```
clear all
```