

Section 4.2 Basis and Dimension

We have talked quite a bit about the idea of span:

Formal Definition: The span of a set of vectors $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\} \in \mathbb{R}^n$ denoted $span\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\}$ is the set of all possible linear combinations

$$c_1 \vec{u}_1 + c_2 \vec{u}_2 + \dots + c_m \vec{u}_m$$

of those vectors, where c_i is any real number.

We have also shown that the span of a set of vectors $span\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\}$ forms a subspace

If we define $S = span\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\}$ with $\vec{u}_i \in \mathbb{R}^n$ Then S is a subspace of $\mathbb{R}^n$.

We also know that sometimes a vector in a set $S = \{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\}$ is redundant and adds nothing to the span of the vectors. That is sometimes one of the vectors in the set is a linear combination of the others.

Quick example: Let $B = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\}$ Because $\begin{bmatrix} 2 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ it is already in the $span\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$.

In fact, $span\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \end{bmatrix} \right\} = span\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. So $\begin{bmatrix} 2 \\ 2 \end{bmatrix}$ adds nothing to the set. Further, after removing it, the remaining two vectors are linearly independent, meaning we need at least those to span the same space.

So... We know the span of a set of vectors is a sub space and that there is a minimal set of vectors, the linearly independent ones, that will span that sub space.

This might prompt us to ask the question, given a subspace S , is there a minimal set of vectors B such that,

$$S = span\{B\}?$$

The answer is yes and we call that set a **basis** for S

A set $B = \{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\}$ is called a basis for a sub space S if

- 1) B spans S , or equivalently $S = span\{B\}$
- 2) B is linearly independent

Section 4.2 Basis and Dimension

Bases are not unique!

Show that $B_1 = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ and $B_2 = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ both span $\mathbb{R}^2$

Let $B = \{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\}$ be a basis for a subspace S . For every vector $\vec{s} \in S$ there exists a unique set of scalars $s_1, s_2, \dots, s_m$ such that the linear combination

$$s_1 \vec{u}_1 + s_2 \vec{u}_2 + \dots + s_m \vec{u}_m = \vec{s}$$

Proof (a quiz problem of yours for next week)

Section 4.2 Basis and Dimension

Let $S = \{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\}$ form a matrix A by putting the vectors in S to be the **rows of A** .

$$A = \begin{bmatrix} -\vec{u}_1 - \\ -\vec{u}_2 - \\ \vdots \\ -\vec{u}_m - \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{21} & u_{22} & \cdots & u_{2n} \\ & \vdots & & \\ u_{m1} & u_{m2} & \cdots & u_{mn} \end{bmatrix}$$

If B is similar to A (obtained by row operations on A) then the span of the rows of A will be the span of the rows of B .

Why? Well think about what a row operation would do... Taking $2R_1 + R_2$ in this case, because the vectors are rows this time, is the same as taking a linear combination of $2\vec{u}_1 + \vec{u}_2$ and replacing $\vec{u}_2$ with that linear combination. Just another vector in the span of the set.

To see it does not change the span: if $\vec{s}$ was in the span of $\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m$ that means that there is a solution to

$$c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_m\vec{u}_m = \vec{s}$$

So is there a solution to

$$c'_1\vec{u}_1 + c'_2(2\vec{u}_1 + \vec{u}_2) + \cdots + c'_m\vec{u}_m = \vec{s} ?$$

You bet! Take $c'_2 = c_2$ and $c'_1 = c_1 - 2c_2$ and $c'_i = c_i$

$$c'_1\vec{u}_1 + c'_2(2\vec{u}_1 + \vec{u}_2) + \cdots + c'_m\vec{u}_m = (c_1 - 2c_2)\vec{u}_1 + 2c_2\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_m\vec{u}_m = c_1\vec{u}_1 + c_2\vec{u}_2 + \cdots + c_m\vec{u}_m = \vec{s}$$

You might think, this could have replaced a possibly linearly independent vector with one that is a linear combination of the others, but remember we **replaced** $\vec{u}_2$ with this new vector, it is no longer in the list and we would not use it directly in linear combinations anymore. Let's just prove it real quick that if $\vec{u}_2$ was linearly independent then so is the result of a row operation on it.

If $\vec{u}_2$ was linearly independent from the other vectors then we can say that there is no solution to

$$c_1\vec{u}_1 + c_3\vec{u}_3 + \cdots + c_m\vec{u}_m = \vec{u}_2$$

If we replace $\vec{u}_2$ with $2\vec{u}_1 + \vec{u}_2$ we get the equation

$$c_1\vec{u}_1 + c_3\vec{u}_3 + \cdots + c_m\vec{u}_m = 2\vec{u}_1 + \vec{u}_2$$

Is there a solution to this one? Well subtract $2\vec{u}_1$ from both sides.

$$(c_1 - 2)\vec{u}_1 + c_3\vec{u}_3 + \cdots + c_m\vec{u}_m = \vec{u}_2$$

This equation would say that indeed there was a linear combination of the other vectors that could get me to $\vec{u}_2$ but we know there is not because we assumed $\vec{u}_2$ to be linearly independent.

Now if u_2 was linearly dependent, then its row operation replacement will be as well, just start with the same equation above but assume there was a solution. Then add 2 to c_1 and you found the solution for the replacement vector!

Section 4.2 Basis and Dimension

Let $S = \{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\}$ form a matrix A by putting the vectors in S to be the **rows of A** .

$$A = \begin{bmatrix} -\vec{u}_1 - \\ -\vec{u}_2 - \\ \vdots \\ -\vec{u}_m - \end{bmatrix} = \begin{bmatrix} u_{11} & u_{12} & \cdots & u_{1n} \\ u_{21} & u_{22} & \cdots & u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ u_{m1} & u_{m2} & \cdots & u_{mn} \end{bmatrix}$$

If B is similar to A (obtained by row operations on A) then the span of the rows of A will be the span of the rows of B .

We can use this theorem to help us find a basis for the span of a set of vectors. Keep in mind that if the set of vectors is linearly independent then it is already a basis for its span, however if there is one or more vectors in the set that are linear combinations of the others we need only remove those so that we are left with the linearly independent ones.

We can obtain a basis for the span of list of vectors in the following way

1. Make a matrix forming the rows using the vectors of the list.
2. Take yourself all the way to echelon form. **Keep in mind that the span of the rows of the original matrix A will be exactly the same as the span of the rows of your echelon matrix. That is what makes this work.**
3. Two things can happen when you get to echelon. Either you have some rows of all zeros or not. If you have some, then you had some linearly dependent vectors in your set. You discovered them by taking a bunch of linear combinations of them (by doing row operations) and found that at some point you had a vector in your set that was a multiple of another, that's how you got zeros.
4. Any row that is not only zeros is a vector in your basis, toss the zero rows away, the remaining ones are linearly independent.

So in short, given a set of vectors $S = \{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_m\}$ make a matrix $A = \begin{bmatrix} -\vec{u}_1 - \\ -\vec{u}_2 - \\ \vdots \\ -\vec{u}_m - \end{bmatrix}$ take A to echelon

form via row operations and throw away any rows that are all zero. Keep all rows, **from the echelon matrix**, that are not all zero, those rows are your basis vectors and are linearly independent.

Section 4.2 Basis and Dimension

```
octave:6> u1=[ 1 3 2 -4]
```

```
u1 =
```

```
1  3  2 -4
```

```
octave:7> u2 = [1 4 -1 2]
```

```
u2 =
```

```
1  4 -1  2
```

```
octave:8> u3=2*u1-4*u2
```

```
u3 =
```

```
-2 -10  8 -16
```

```
octave:9> u4=3*u3-2*u2
```

```
u4 =
```

```
-8 -38 26 -52
```

```
octave:10> A=[u1 ; u2 ; u3 ; u4]
```

```
A =
```

```
1  3  2 -4
1  4 -1  2
-2 -10  8 -16
-8 -38 26 -52
```

```
octave:11> rref(A)
```

```
ans =
```

```
1  0 11 -22
0  1 -3  6
0  0  0  0
0  0  0  0
```

Section 4.2 Basis and Dimension

Let $U = [\vec{u}_1 \cdots \vec{u}_m]$ and $V = [\vec{v}_1 \cdots \vec{v}_m]$ are two equivalent matrices, that is they can be transformed into one another via row operations. Then any linear dependence that exist among $\vec{u}_1, \dots, \vec{u}_m$ also exists between $\vec{v}_1 \cdots \vec{v}_m$. **Note: here, the matrices were formed by putting vectors as columns**

Why? Because we can use pivot positions to identify which vectors are linearly independent.

```
octave:14> A=[u1' , u2' , u3' ,u4',zeros(4,1) ]
```

A =

1	1	-2	-8	0
3	4	-10	-38	0
2	-1	8	26	0
-4	2	-16	-52	0

```
octave:15> rref(A)
```

ans =

1	0	2	6	-0
0	1	-4	-14	0
0	0	0	0	0
0	0	0	0	0

let $x_4 = s, x_3 = t$ then $x_2 = 4t + 14s, x_1 = -2t - 6$ More than the trivial solution works.

But what if we had only looked at the two vectors which have pivot positions.....

```
octave:16> B=[A(:,1) , A(:,2) , A(:,5)]
```

B =

1	1	0
3	4	0
2	-1	0
-4	2	0

```
octave:17> rref(B)
```

ans =

1	0	-0
0	1	0
0	0	0
0	0	0

Now there are no free variables... so the only solution here is $x_1 = x_2 = 0$ so the first two columns of A were linearly independent.

Section 4.2 Basis and Dimension

```
octave:14> A=[u1' , u2' , u3' ,u4',zeros(4,1) ]  
A =
```

```
1   1   -2   -8   0  
3   4  -10  -38   0  
2  -1    8   26   0  
-4   2  -16  -52   0
```

Remember that removing linearly dependent vectors from the list does not change the span so

$$\text{span} \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -1 \\ 2 \end{bmatrix} \right\} = \text{span} \left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ -10 \\ 8 \\ -16 \end{bmatrix}, \begin{bmatrix} -8 \\ -38 \\ -26 \\ -52 \end{bmatrix} \right\}$$

And since $\left\{ \begin{bmatrix} 1 \\ 3 \\ 2 \\ -4 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -1 \\ 2 \end{bmatrix} \right\}$ are linearly independent, they are a basis!

So given a set of vectors you can find a basis for their span by throwing them into a matrix as columns, going to rref form and tossing any vectors from the **original matrix** that don't have pivot positions.

Now that we know how to throw away useless vectors to get a basis, it is time to state the following theorem.

If S is a subspace of R^n , then every basis of S has the same number of vectors.

Section 4.2 Basis and Dimension

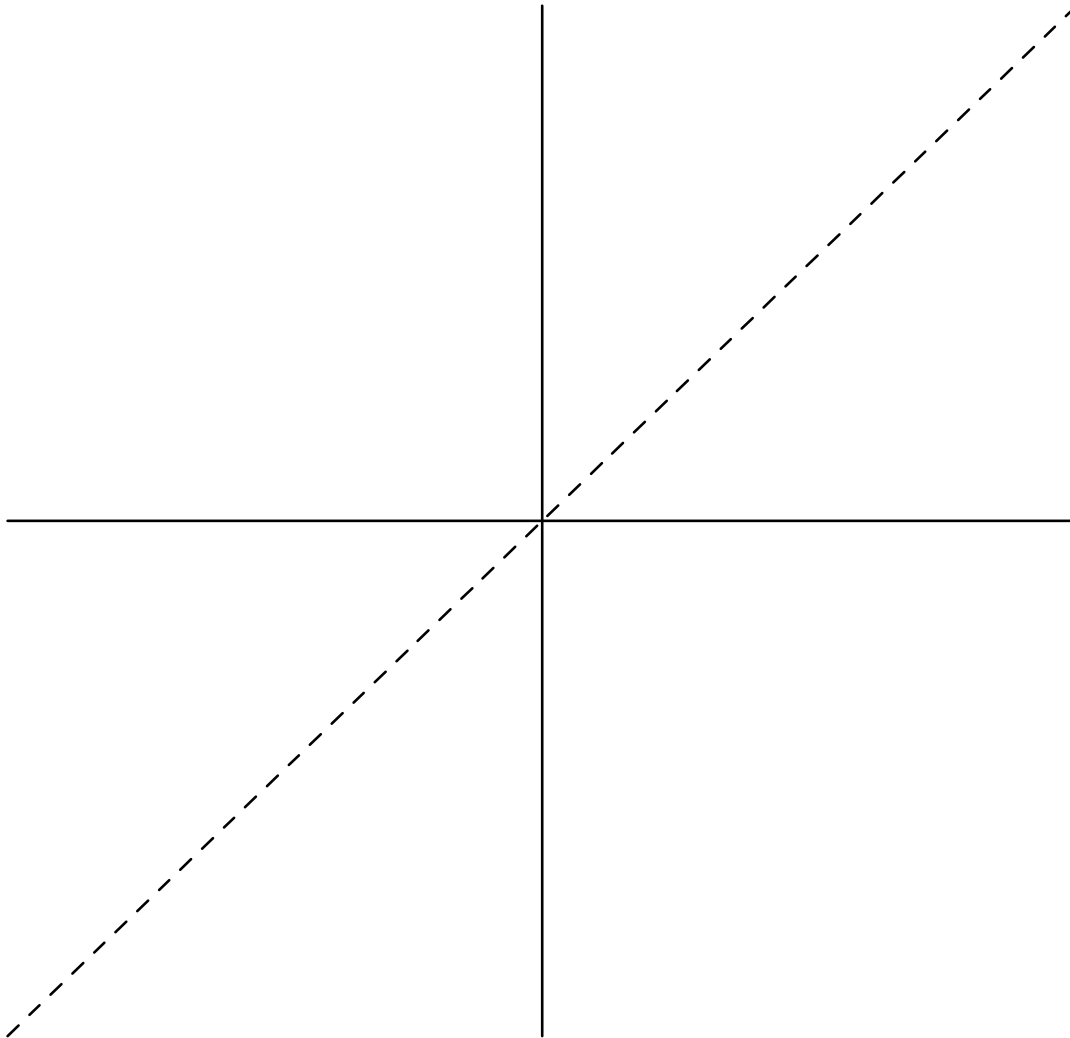
If S is a subspace of R^n , then every basis of S has the same number of vectors.

We will prove this for a simple case of $S_1 = \{u_1, u_2\}$ and $S_2 = \{v_1, v_2, v_3\}$

Section 4.2 Basis and Dimension

Let S be a subspace in $\mathbb{R}^n$ with basis B then the dimension $\dim(S)$ is the number of basis vectors in B .

Example: What is the dimension of $B = \left\{ \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$?



Section 4.2 Basis and Dimension

The natural basis for R^n

The vectors

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \dots e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots e_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Where $e_i \in R^n$ form a basis for R^n

What is a quick basis for R^2 ?

What is a quick basis for R^{10} ?

Section 4.2 Basis and Dimension

The Unifying Theorem Expanded again!

Let $S = \{\vec{a}_1, \dots, \vec{a}_n\}$ be a set of n vectors in R^n , let $A = [\vec{a}_1 \ \vec{a}_2 \ \dots \ \vec{a}_n]$. Let $T: R^n \rightarrow R^n$ be given by $T(\vec{x}) = A\vec{x}$. Then the following are equivalent:

- (a) S spans R^n
- (b) S is linearly independent
- (c) $Ax = b$ has a unique solution for all $\vec{b} \in R^n$
- (d) T is onto
- (e) T is one – to – one
- (f) A is invertable
- (g) $\ker(T)=0$ or equivalently the Nullspace(A)=0
- (h) S is a basis for R^n**