

Section 4.3 Row and Column Space

Let A be an $n \times m$ matrix

- (a) The row space of A is the subspace of R^m spanned by the rows of A
- (b) The column space of A is the subspace of R^n spanned by the columns of A

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 10 & 11 & 12 \end{bmatrix}$$

$\text{row}(A)$

$\text{Span} \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}, \begin{bmatrix} 7 \\ 8 \\ 9 \end{bmatrix}, \begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix} \right\}$

$\text{col}(A)$

$\text{Span} \left\{ \begin{bmatrix} 1 \\ 4 \\ 7 \\ 10 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \\ 11 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 9 \\ 12 \end{bmatrix} \right\}$

Let A be a matrix and B be an echelon or reduced row echelon form of A , then:

- (a) The nonzero rows of B form a basis for $\text{row}(A)$.
- (b) The columns of A corresponding to the pivot columns in B form a basis for $\text{col}(A)$.

octave:8> A=[1,2,3;4,5,6;7,8,9;10,11,12]
A =

1	2	3
4	5	6
7	8	9
10	11	12

octave:9> rref(A)
ans =

1	0	-1
0	1	2
0	0	0
0	0	0

$B_{\text{col}} = \text{Span} \left\{ \begin{bmatrix} 1 \\ 4 \\ 7 \\ 10 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \\ 8 \\ 11 \end{bmatrix} \right\}$ What is its dimension?

$\text{Dim}(\text{row}(A)) = 2$

$\text{Dim}(\text{col}(A)) = 2$

$B_{\text{row}} = \left\{ \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$

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For any matrix of A the dimension of the row space equals the dimension of the column space.

This is because the number of columns with pivot positions will equal the number of rows with pivot positions! In rref form the rows without pivot positions will all be zero and the number of linearly independent vectors in the row space is the number of nonzero rows. For the column space the columns with pivots corresponded to the linearly independent vectors in the original matrix.

```
octave:8> A=[1,2,3;4,5,6;7,8,9;10,11,12]
A =
```

1	2	3
4	5	6
7	8	9
10	11	12

```
octave:9> rref(A)
ans =
```

→	↓	↓	
→	1	0	-1
→	0	1	2
	0	0	0
	0	0	0

The Rank of a matrix A , $\text{Rank}(A)$, is the dimension of the row or column space, which is the same as the number of pivot positions in rref form

What is the rank?

```
octave:8> A=[1,2,3;4,5,6;7,8,9;10,11,12]
A =
```

1	2	3
4	5	6
7	8	9
10	11	12

```
octave:9> rref(A)
ans =
```

1	0	-1
0	1	2
0	0	0
0	0	0

$\text{Rank}(A) = 2$

```
octave:14> A=[ 1 4 2 3 2; 2 2 10 29 3; 1 2 3 -1 4]
A =
```

1	4	2	3	2
2	2	10	29	3
1	2	3	-1	4

```
octave:15> rref(A)
ans =
```

1	0	0	-155/3	46/3
0	1	0	47/6	-13/6
0	0	1	35/3	-7/3

3 pivots
 $\text{Rank}(A) = 3$
 $\text{Dim}(\text{Row}) = 3$
 $\text{Dim}(\text{Col}) = 3$

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The nullity of a matrix A , $\text{null}(A)$, is the dimension of the null space.

Let's find the null space

octave:2> A=[-1 0 0 4 5 4 0; 6 2 1 2 5 0 0; 3 2 -5 -1 0 4 0]

A = $\begin{bmatrix} v_1 & v_2 & v_3 & v_4 & v_5 & v_6 \\ -1 & 0 & 0 & 4 & 5 & 4 \\ 6 & 2 & 1 & 2 & 5 & 0 \\ 3 & 2 & -5 & -1 & 0 & 4 \end{bmatrix} \begin{matrix} 0 \\ 0 \\ 0 \end{matrix}$ $\rightarrow A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = 0 \rightarrow x_1 \vec{v}_1 + x_2 \vec{v}_2 + x_3 \vec{v}_3 + x_4 \vec{v}_4 + x_5 \vec{v}_5 + x_6 \vec{v}_6 = 0$

octave:3> format rat

octave:4> rref(A)

ans =

	x_1	x_2	x_3	x_4	x_5	x_6	
$\rightarrow$	1	0	0	-4	-5	-4	0
	0	$\frac{1}{0}$	0	$\frac{47}{4}$	$\frac{95}{6}$	$\frac{34}{3}$	0
	0	0	1	$\frac{5}{2}$	$\frac{10}{3}$	$\frac{4}{3}$	0

$x_4 = t_3 \quad x_5 = t_2 \quad x_6 = t_1$

$x_3 + \frac{5}{2}t_3 + \frac{10}{3}t_2 + \frac{4}{3}t_1 = 0 \rightarrow x_3 = -\frac{5}{2}t_3 - \frac{10}{3}t_2 - \frac{4}{3}t_1$

$x_2 + \frac{47}{4}t_3 + \frac{95}{6}t_2 + \frac{34}{3}t_1 = 0 \rightarrow x_2 = -\frac{47}{4}t_3 - \frac{95}{6}t_2 - \frac{34}{3}t_1$

$x_1 = 4t_3 + 5t_2 + 4t_1$

$\downarrow$

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 4t_3 + 5t_2 + 4t_1 \\ -\frac{47}{4}t_3 - \frac{95}{6}t_2 - \frac{34}{3}t_1 \\ -\frac{5}{2}t_3 - \frac{10}{3}t_2 - \frac{4}{3}t_1 \\ t_3 \\ t_2 \\ t_1 \end{bmatrix} = t_1 \begin{bmatrix} 4 \\ -\frac{34}{3} \\ -\frac{4}{3} \\ 0 \\ 0 \\ 1 \end{bmatrix} + t_2 \begin{bmatrix} 5 \\ -\frac{95}{6} \\ -\frac{10}{3} \\ 0 \\ 1 \\ 0 \end{bmatrix} + t_3 \begin{bmatrix} 4 \\ -\frac{47}{4} \\ -\frac{5}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$\text{Null}(A) = \text{Span} \left\{ \begin{bmatrix} 4 \\ -\frac{34}{3} \\ -\frac{4}{3} \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -\frac{95}{6} \\ -\frac{10}{3} \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -\frac{47}{4} \\ -\frac{5}{2} \\ 1 \\ 0 \\ 0 \end{bmatrix} \right\} \quad \text{null}(A) = 3$

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Rank-Nullity theorem

Let A be an $n \times m$ matrix then,

$$\text{Rank}(A) + \text{Nullity}(A) = m$$

Proof

octave:4> rref(A)
ans =

3×5

1	0	0	-4	-5	-4	0
0	1	0	47/4	95/6	34/3	0
0	0	1	5/2	10/3	4/3	0

3 Pivots 2 free variable
 $\text{Rank}(A) = 3$ $\text{Null}(A) = 2$ $3 + 2 = 5$

The $\text{Rank}(A)$ equals the number of columns with a pivot.

The $\text{Nullity}(A)$ is the number of basis vectors that span the null space, since each free variable we find generates a basis vector and each free variable comes from a non pivot column the $\text{Nullity}(A)$ is the number of nonpivot columns of A .

Since the number of pivot columns + the number of non pivot columns is the number of columns in A

$$\text{Rank}(A) + \text{Nullity}(A) = m$$

$\downarrow$ $\downarrow$
pivot + non pivot = # columns

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The Unifying Theorem Expanded again!

Let $S = \{\vec{a}_1, \dots, \vec{a}_n\}$ be a set of n vectors in R^n , let $A = [\vec{a}_1 \ \vec{a}_2 \ \dots \ \vec{a}_n]$. Let $T: R^n \rightarrow R^n$ be given by $T(\vec{x}) = A\vec{x}$. Then the following are equivalent:

- (a) S spans R^n
- (b) S is linearly independent
- (c) ~~$Ax = b$~~ has a unique solution for all $\vec{b} \in R^n$
- (d) T is onto
- (e) T is one-to-one
- (f) A is invertable
- (g) $\ker(T)=0$ or equivalently the Nullspace(A)=0
- (h) S is a basis for R^n
- (i) $\text{col}(A) = R^n$
- (j) $\text{row}(A) = R^n$
- (k) ~~$\text{Rank}(A) = n$~~

A is $n \times n$