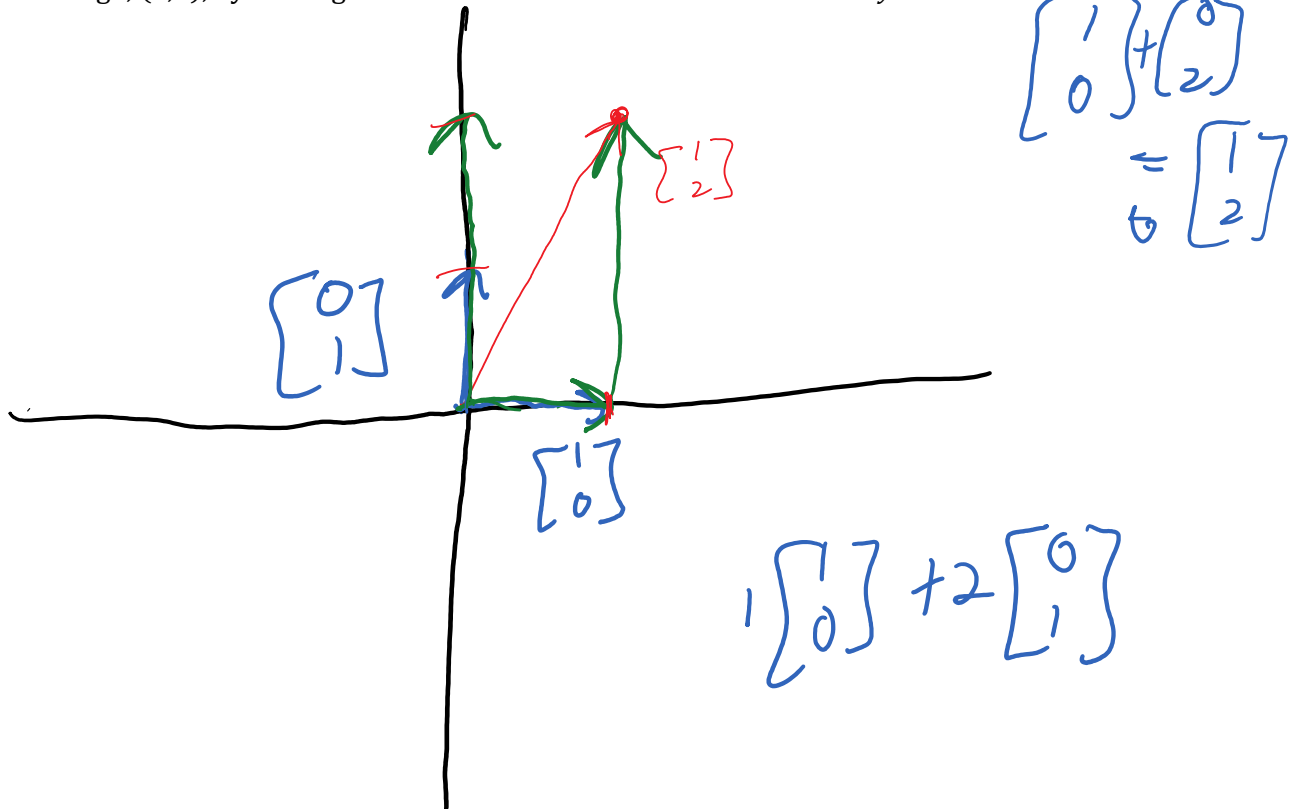


Section 4.4 Change of Basis

In chapter 2 we talked about the geometry of vectors. We thought of the vector $\begin{bmatrix} 1 \\ 2 \end{bmatrix} \in \mathbb{R}^2$ as moving away from the origin, $(0,0)$, by moving 1 unit in the x direction and 2 units in the y direction.



However, now that we have talked about a Basis, we can say what we were really doing, taking a linear combination the basis vectors $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$. That is for the linear combination $c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ the vector $\vec{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ meant that $c_1 = 1, c_2 = 2$ so that the combination

$$1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \text{after } +4, 5$$

took us to the point $(1,2)$ in R^2 .

we pointed to $(1, 2)$

It just so happens that in this particular case, $c_1 = 1$ and $c_2 = 2$ took us to the point (1,2) What if we wanted to get to the point (1,2) in space by taking a linear combination of a differet set of basis vectors, say $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\}$? That is, what are c_1, c_2 in $\vec{v} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ so that the linear combination

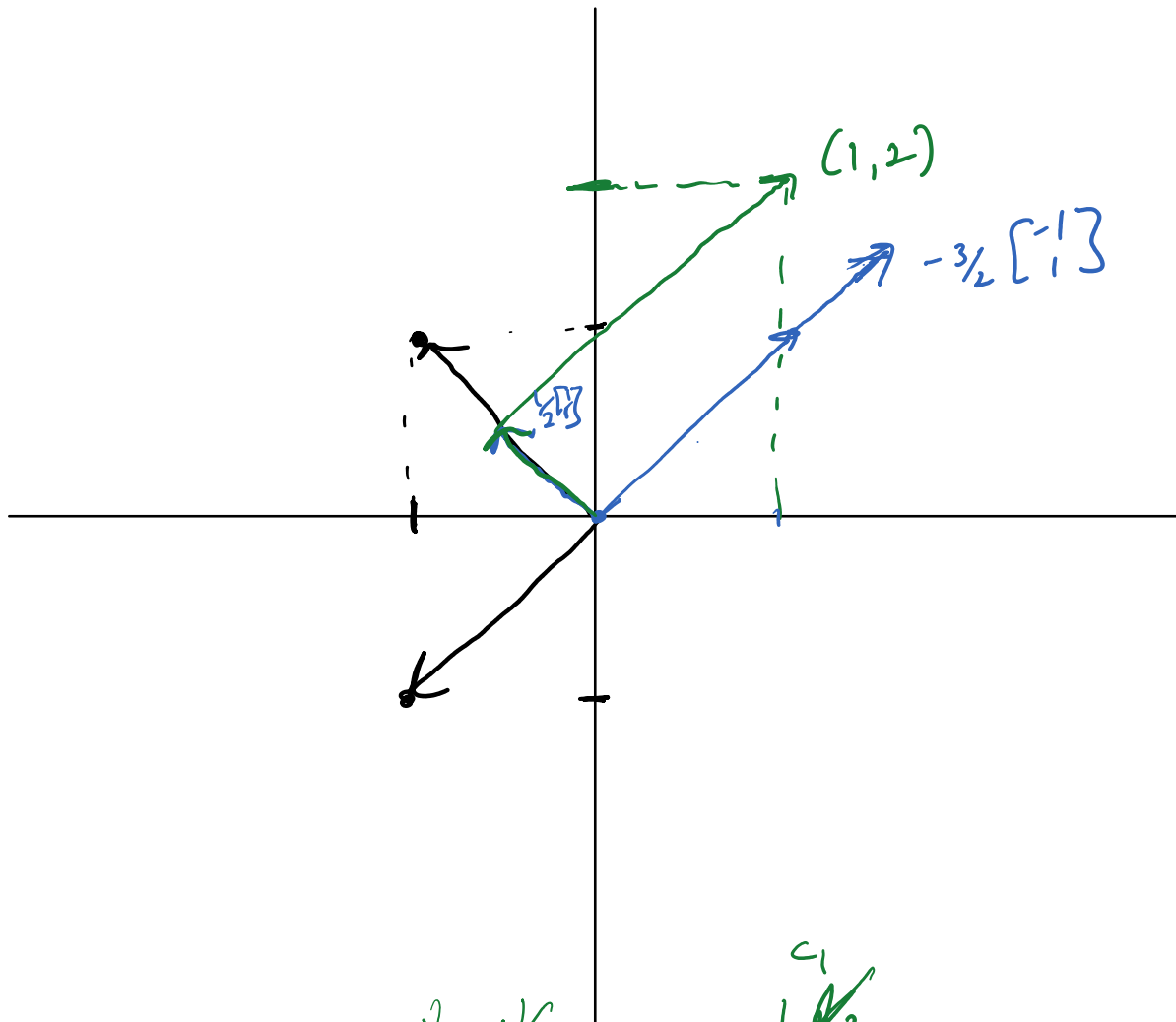
$$c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

takes us to the point $(1,2)$ in R^2 ?

Section 4.4 Change of Basis

$$c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

In this case we can take $c_1 = \frac{1}{2}$ and $c_2 = -\frac{3}{2} = -1.5$



So with respect to the basis $B_2 = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\}$ the vector $\vec{v} = \begin{bmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{bmatrix}$ gives us c_1, c_2 so that the

linear combination $\frac{1}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ geometrically takes us to the point (1,2) of course if we do the vector addition

$$\frac{1}{2} \begin{bmatrix} -1 \\ 1 \end{bmatrix} - \frac{3}{2} \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} + \frac{3}{2} \\ \frac{1}{2} + \frac{3}{2} \end{bmatrix} = \begin{bmatrix} \frac{2}{2} \\ \frac{4}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Which makes the notation very confusing! So instead we would denote our vector this way:

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}_{B_2} = \begin{bmatrix} \frac{1}{2} \\ -\frac{3}{2} \end{bmatrix}$$

Section 4.4 Change of Basis

Let's pick this apart a bit, first we need to establish the standard basis, it is the one to which we will reference all others.

The standard, or natural, basis for R^n is the set of n vectors $\{\vec{e}_1, \vec{e}_2, \dots, \vec{e}_n\}$, $\vec{e}_i \in R^n$ where

$$e_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix}, e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \dots, e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, e_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

With this basis, the coefficients of the linear combination of basis vectors that take us to the point $(x_1, x_2, \dots, x_n)$ are

stored in the vector $\vec{u} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, so that the vector corresponds to the point!

$$x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix} + \dots + x_i \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \dots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{bmatrix}$$

The natural basis for R^2 is $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$, note that it is **not** $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$. The order here matters since for the basis

u_1, u_2
 $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ to get to $(1,2)$ the vector with coefficients of the LC would be $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ $\begin{matrix} \rightarrow c_1 \\ \rightarrow c_2 \end{matrix}$

$$1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

u_1, u_2
 But for $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ to get to $(1,2)$ the vector with coefficients would be $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $\begin{matrix} \leftarrow c_1 \\ \leftarrow c_2 \end{matrix}$

$$2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Section 4.4 Change of Basis

Back to picking apart the notation

1. The vector in the standard basis, which corresponds to the point (1,2) in space,

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix}_{B_2} = \begin{bmatrix} 1 \\ \frac{1}{2} \\ 3 \\ -\frac{3}{2} \end{bmatrix}$$

3. using the coefficients $c_1 = \frac{1}{2}, c_2 = -\frac{3}{2}$.

$$2. B_2 = \left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\}$$

can be represented by a linear combination of these basis vectors

$$c_1 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

which themselves are represented with respect to the standard basis,

Suppose you want to geometrically understand the vector $[\vec{u}]_B = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ where $B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}$

what point in R^3 does it point to?

Let's think about what this says, it says that the vector the form of $\vec{u}$ in the standard basis can be represented by the linear combination

$$3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \\ 5 \end{bmatrix} \text{ so it points to } (7, -5, 5)$$

$$\text{So in the standard basis } \vec{u} = \begin{bmatrix} 7 \\ -5 \\ 5 \end{bmatrix} = 7 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 5 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 5 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Section 4.4 Change of Basis

Note that we can represent this with matrix multiplication

$$[\vec{u}]_B = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \text{ where } B = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}$$

$$3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \\ 5 \end{bmatrix}$$

In fact the action of the matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & 1 \\ 1 & 0 & 2 \end{bmatrix}$ on the vector $[\vec{u}]_B = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ was to convert it to the vector $\vec{u} = \begin{bmatrix} 7 \\ -5 \\ 5 \end{bmatrix}$. It can be viewed as a change of basis if we like.

Convert the coordinate vector $[\mathbf{x}]_B$ from the given basis B to the standard basis.

$$B = \left\{ \begin{bmatrix} -8 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ -3 \end{bmatrix} \right\}, [\mathbf{x}]_B = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

1a

$$\mathbf{x} = \begin{bmatrix} \text{ } \\ \text{ } \end{bmatrix}$$

 [34; -11]

$$\vec{x} = -2 \begin{bmatrix} -8 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 6 \\ -3 \end{bmatrix}$$

Standard basis

```
octave:2> format rat
octave:3> A=[-8 6; 1 -3]
A =
```

$$\begin{bmatrix} -8 & 6 \\ 1 & -3 \end{bmatrix}$$

```
octave:4> xB=[-2; 3]
xB =
```

$$\begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

So to go from the basis B to the standard basis, form a matrix $A = [\vec{b}_1 \ \vec{b}_2 \ \dots \ \vec{b}_n]$ and multiply the vector you wish to convert!

```
octave:5> A*xB
ans =
```

$$\begin{bmatrix} 34 \\ -11 \end{bmatrix} = \vec{x}$$

Let $\vec{x}$ be expressed with respect to the standard basis, and let $B = \{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\}$ be any basis of $\mathbb{R}^n$.

If $U = [\vec{u}_1 \ \vec{u}_2 \ \dots \ \vec{u}_n]$ then

(a) $\vec{x} = U[\vec{x}]_B$

(b) $[\vec{x}]_B = U^{-1}\vec{x}$

$$U^{-1}\vec{x} = U^{-1}U[\vec{x}]_B \quad U^{-1}\vec{x} = [\vec{x}]_B$$

Section 4.4 Change of Basis

What if we want to go from a basis B_1 to a basis B_2 for which neither is the standard basis. Its easy! We just go from one to the standard basis and then from the standard basis to the other.

Let $[\vec{u}]_{B_1} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$ where $B_1 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} \right\}$ find $[\vec{u}]_{B_2}$ where $B_2 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix} \right\}$

Here is how it works, We could go from B_1 to the standard basis by:

$$\begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ -5 \\ 5 \end{bmatrix}$$

Or back from the standard basis to the B_1 by:

$$\begin{bmatrix} -6 & -2 & 7 \\ 1 & 0 & -1 \\ 3 & 1 & -3 \end{bmatrix} \begin{bmatrix} 7 \\ -5 \\ 5 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

$U^{-1} \quad \vec{v} \quad [\vec{v}]_{B_1}$

Where $\begin{bmatrix} -6 & -2 & 7 \\ 1 & 0 & -1 \\ 3 & 1 & -3 \end{bmatrix} = \left(\begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & 1 \\ 1 & 0 & 2 \end{bmatrix} \right)^{-1}$

We could also go from a vector in from B_2 to the standard basis by

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}_{B_2} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

Or back from the standard basis to the B_2 by:

$$\begin{bmatrix} -1 & 1 & 1 \\ 2 & -2 & -1 \\ 2 & -3 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}_{B_2}$$

Where $\begin{bmatrix} -1 & 1 & 1 \\ 2 & -2 & -1 \\ 2 & -3 & -1 \end{bmatrix} = \left(\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -1 \\ 2 & 1 & 0 \end{bmatrix} \right)^{-1}$

So we could go from $[\vec{u}]_{B_1}$ to $\vec{u}$ to $[\vec{u}]_{B_2}$ by

$$\begin{bmatrix} -1 & 1 & 1 \\ 2 & -2 & -1 \\ 2 & -3 & -1 \end{bmatrix} \left(\begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -1 & 1 & 1 \\ 2 & -2 & -1 \\ 2 & -3 & -1 \end{bmatrix} \begin{bmatrix} 7 \\ -5 \\ 5 \end{bmatrix} = \begin{bmatrix} -7 \\ 19 \\ 24 \end{bmatrix} = [\vec{u}]_{B_2}$$

Or we could multiply the matrices together first to get a change of basis matrix from B_1 to B_2

$$\begin{bmatrix} -1 & 1 & 1 \\ 2 & -2 & -1 \\ 2 & -3 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 \\ 0 & -3 & 1 \\ 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & -4 & 1 \\ 1 & 8 & 0 \\ 1 & 11 & -1 \end{bmatrix}$$

That is $\begin{bmatrix} 0 & -4 & 1 \\ 1 & 8 & 0 \\ 1 & 11 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -7 \\ 19 \\ 24 \end{bmatrix}$

$\uparrow$
change of basis matrix

Section 4.4 Change of Basis

Let $B_1 = \{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\}$ and $B_2 = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ be bases for R^n . Let $U = [\vec{u}_1 \ \vec{u}_2 \ \dots \ \vec{u}_n]$ and $V = [\vec{v}_1 \ \vec{v}_2 \ \dots \ \vec{v}_n]$

$$(a) [\vec{x}]_{B_2} = V^{-1}U[\vec{x}]_{B_1}$$

$$(b) [\vec{x}]_{B_1} = U^{-1}V[\vec{x}]_{B_2}$$

Changing Bases in Subspaces. Sometimes the number of basis vectors is less than the dimension of the space to which they belong. That is our basis vectors might belong to say R^3 , however maybe there are only two of them spanning a plane. In this case it is a bit harder to find the change of basis matrix as it will not just be made of the basis vectors themselves.

Find the change of basis matrix from B_1 to B_2 .

$$B_1 = \left\{ \begin{bmatrix} 2 \\ -5 \\ 4 \end{bmatrix}, \begin{bmatrix} -5 \\ 6 \\ -7 \end{bmatrix} \right\}, B_2 = \left\{ \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix} \right\}$$

First let's understand what a vector $\vec{x}$ in the subspace spanned by B_1 but expressed with respect to the standard basis would look like with respect to B_1

Let's pick $\vec{x} = 2 \begin{bmatrix} 2 \\ -5 \\ 4 \end{bmatrix} + 1 \begin{bmatrix} -5 \\ 6 \\ -7 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix}$ which is clearly in the span of B_1 so with respect to the basis $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} x =$

$\begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix}$ since we point to the point $(-1, -4, 1)$ in R^3 by taking the linear combination $-1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} - 4 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. However

with respect to the basis B_1 the linear combination was $2 \begin{bmatrix} 2 \\ -5 \\ 4 \end{bmatrix} + 1 \begin{bmatrix} -5 \\ 6 \\ -7 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix}$ to point to the point $(-1, -4, 1)$ so

$\begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix}_{B_1} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. So here it is defined by two numbers, remember B_1 defines a plane of dimension 2!

The same vector with respect to B_2 is $\begin{bmatrix} 0 \\ -1 \end{bmatrix}$ since $0 \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix}$ so $\begin{bmatrix} -1 \\ -4 \\ 1 \end{bmatrix}_{B_2} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

So a change of basis matrix between the two bases would be 2×2 !

$$\begin{bmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$$

Section 4.4 Change of Basis

The trick here is to find out how to represent the basis vectors of B_2 in terms of B_1 . That is we want to find $c_{11}, c_{21}, c_{12}, c_{22}$ such that

$$\begin{bmatrix} 2 \\ -5 \\ 4 \end{bmatrix} = c_{11} \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix} + c_{21} \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix} \quad \text{and}$$

$$\begin{bmatrix} -5 \\ 6 \\ -7 \end{bmatrix} = c_{12} \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix} + c_{22} \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 \\ -1 & 4 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c_{11} \\ c_{21} \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 \\ -1 & 4 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} c_{12} \\ c_{22} \end{bmatrix} = \begin{bmatrix} -5 \\ 6 \\ -7 \end{bmatrix}$$

$$\left[\begin{array}{cc|c} 3 & 1 & 2 \\ -1 & 4 & -5 \\ 3 & -1 & 4 \end{array} \right]$$

$$\left[\begin{array}{cc|c} 3 & 1 & -5 \\ -1 & 4 & 6 \\ 3 & -1 & -7 \end{array} \right]$$

```
octave:21> A=[3 1; -1 4; 3 -1;]
```

```
A =
```

$$\begin{bmatrix} 3 & 1 \\ -1 & 4 \\ 3 & -1 \end{bmatrix}$$

```
octave:22> [A, [2;-5;4]]
```

```
ans =
```

$$\begin{bmatrix} 3 & 1 & 2 \\ -1 & 4 & -5 \\ 3 & -1 & 4 \end{bmatrix}$$

```
octave:23> rref([A, [2;-5;4]])
```

```
ans =
```

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$c_{11} = 1, c_{21} = -1$$

```
octave:18> A=[3 1; -1 4; 3 -1;]
```

```
A =
```

$$\begin{bmatrix} 3 & 1 \\ -1 & 4 \\ 3 & -1 \end{bmatrix}$$

```
octave:19> [A, [-5;6;-7]]
```

```
ans =
```

$$\begin{bmatrix} 3 & 1 & -5 \\ -1 & 4 & 6 \\ 3 & -1 & -7 \end{bmatrix}$$

```
octave:20> rref([A, [-5;6;-7]])
```

```
ans =
```

$$\begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$c_{12} = -2, c_{22} = 1$$

So, $\left[\begin{bmatrix} 2 \\ -5 \\ 4 \end{bmatrix} \right]_{B_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ since,

$$1 \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix} - 1 \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -5 \\ 4 \end{bmatrix}$$

and

$\left[\begin{bmatrix} -5 \\ 6 \\ -7 \end{bmatrix} \right]_{B_1} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ since,

$$-2 \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix} = \begin{bmatrix} -5 \\ 6 \\ -7 \end{bmatrix}$$

Section 4.3 Change of Basis

$$\left[\begin{bmatrix} 2 \\ -5 \\ 4 \end{bmatrix} \right]_{B_2} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\left[\begin{bmatrix} -5 \\ 6 \\ -7 \end{bmatrix} \right]_{B_2} = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Ok so now let $\vec{x} = a_1 \begin{bmatrix} 2 \\ -5 \\ 4 \end{bmatrix} + a_2 \begin{bmatrix} -5 \\ 6 \\ -7 \end{bmatrix}$ be some vector in our subspace. Then we can say

$[\vec{x}]_{B_1} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$. Now suppose $[\vec{x}]_{B_2} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$ so that $\vec{x} = d_1 \begin{bmatrix} 3 \\ -1 \\ 3 \end{bmatrix} + d_2 \begin{bmatrix} 1 \\ 4 \\ -1 \end{bmatrix}$ for the same $\vec{x}$

Then

$$\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = [\vec{x}]_{B_2} = \left[a_1 \begin{bmatrix} 2 \\ -5 \\ 4 \end{bmatrix} + a_2 \begin{bmatrix} -5 \\ 6 \\ -7 \end{bmatrix} \right]_{B_2} = a_1 \left[\begin{bmatrix} 2 \\ -5 \\ 4 \end{bmatrix} \right]_{B_2} + a_2 \left[\begin{bmatrix} -5 \\ 6 \\ -7 \end{bmatrix} \right]_{B_2} = a_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} + a_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} [\vec{x}]_{B_1}$$

So $\begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix} [\vec{x}]_{B_1} = [\vec{x}]_{B_2}$ and thus $\begin{bmatrix} 1 & -2 \\ -1 & 1 \end{bmatrix}$ is our change of basis matrix which can be made by taking the columns to be the representation of the basis vectors in B_1 in B_2

We have a general theorem about this.

Let S be a subspace of R^n with bases $B_1 = \{u_1, \dots, u_k\}$ and $B_2 = \{v_1, \dots, v_k\}$, then the change of base matrix from B_1 to B_2 is given by:

$$C = \left[[\vec{u}_1]_{B_2} \quad [\vec{u}_2]_{B_2} \quad \dots \quad [\vec{u}_k]_{B_2} \right]$$

So to find the change of basis matrix from B_1 to B_2 you can just find the representation of each of the basis vectors in B_1 in B_2 and take those as columns to form the matrix.

Section 4.4 Change of Basis

A Score Booster Problem!

If you are able to correctly do this mini assignment and turn it in the Tuesday after Exam 1 5 points will be added to your exam score up to 100%.

You are an astronomer who is looking at a star named AOC1337. You have just been given measurements of two radio frequencies ν_1 and ν_2 . According to steller physics the relationship between the the intensities of the two frequencies I_{ν_1}, I_{ν_2} is

$$I_{\nu_2} = aI_{\nu_1} + b$$

where the values of (a, b) can be used to classify the star.

A NASA satellite measures the frequencies for 5 days and obtains the set of values of the intensities as

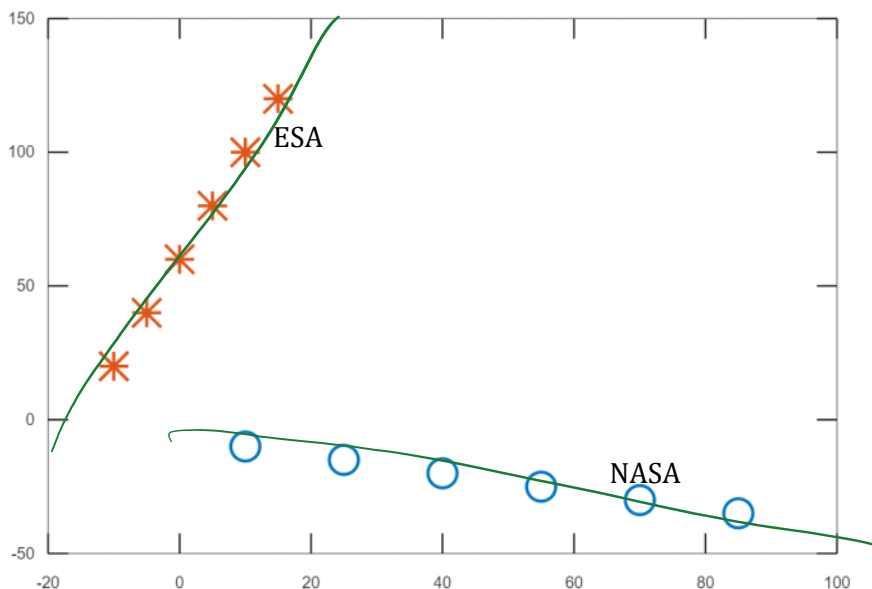
$$I_{\nu_1} = [10 \quad 25 \quad 40 \quad 55 \quad 70 \quad 85]$$

$$I_{\nu_2} = [-10 \quad -15 \quad -20 \quad -25 \quad -30 \quad -35]$$

However an ESA satellite measured the frequencies intensities for the same five days and found

$$I_{\nu_1} = [-10 \quad -5 \quad 0 \quad 5 \quad 10 \quad 15]$$

$$I_{\nu_2} = [20 \quad 40 \quad 60 \quad 80 \quad 100 \quad 120]$$



Section 4.4 Change of Basis

Clearly this data is horribly inconsistent. Using a well-known star you find that for measurements I_{v_1}, I_{v_2} that the NASA satellite has an instrument error given by

$$\left\{ \begin{array}{l} I_{v_1} = I_{v_1}^{true} + I_{v_2}^{true} \\ I_{v_2} = -I_{v_2}^{true} \end{array} \right\}$$

Where as for the ESA satellite you find it has instrument error given by,

$$\left\{ \begin{array}{l} I_{v_1} = I_{v_1}^{true} - I_{v_2}^{true} \\ I_{v_2} = I_{v_1}^{true} + 2 I_{v_2}^{true} \end{array} \right\}$$

We can think of the errors in terms of bases, we would say that the NASA and ESA satellites are expressing the true measurement, which would be in the standard basis, with respect to two different bases, that is

$$\begin{bmatrix} I_{v_1}^{true} \\ I_{v_2}^{true} \end{bmatrix} = \begin{bmatrix} I_{v_1}^{true} \\ I_{v_2}^{true} \end{bmatrix}_{NASA} = \begin{bmatrix} I_{v_1}^{true} \\ I_{v_2}^{true} \end{bmatrix}_{ESA}$$

1. From the equations giving the errors above, find the bases for the NASA and ESA satellites

$$NASA = \left\{ \begin{bmatrix} ? \\ ? \end{bmatrix}, \begin{bmatrix} ? \\ ? \end{bmatrix} \right\} \quad ESA = \left\{ \begin{bmatrix} ? \\ ? \end{bmatrix}, \begin{bmatrix} ? \\ ? \end{bmatrix} \right\}$$

2. Find the change of basis matrices in the equations below

$$\begin{array}{ll} \underline{A} \begin{bmatrix} I_{v_1}^{true} \\ I_{v_2}^{true} \end{bmatrix}_{NASA} = \begin{bmatrix} I_{v_1}^{true} \\ I_{v_2}^{true} \end{bmatrix} & \underline{B} \begin{bmatrix} I_{v_1}^{true} \\ I_{v_2}^{true} \end{bmatrix}_{ESA} = \begin{bmatrix} I_{v_1}^{true} \\ I_{v_2}^{true} \end{bmatrix} \\ \underline{C} \begin{bmatrix} I_{v_1}^{true} \\ I_{v_2}^{true} \end{bmatrix}_{NASA} = \begin{bmatrix} I_{v_1}^{true} \\ I_{v_2}^{true} \end{bmatrix}_{ESA} & \underline{D} \begin{bmatrix} I_{v_1}^{true} \\ I_{v_2}^{true} \end{bmatrix}_{ESA} = \begin{bmatrix} I_{v_1}^{true} \\ I_{v_2}^{true} \end{bmatrix}_{NASA} \end{array}$$

3. Use the matrices A and B to correct the NASA and ESA data, use the resulting values to find (a, b) for the equation.

$$I_{v_2} = aI_{v_1} + b$$

Using the data given at the beginning. Plot the corrected data for each satellite. Use the next page to turn in your answers.

Score Booster Exam 1

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Name

PID

Problem 1

$$NASA = \left\{ \begin{bmatrix} \square \\ \square \end{bmatrix}, \begin{bmatrix} \square \\ \square \end{bmatrix} \right\}$$

$$ESA = \left\{ \begin{bmatrix} \square \\ \square \end{bmatrix}, \begin{bmatrix} \square \\ \square \end{bmatrix} \right\}$$

Problem 2

$$A = \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} \quad B = \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} \quad C = \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix} \quad D = \begin{bmatrix} \square & \square \\ \square & \square \end{bmatrix}$$

Problem 3.

$$a =$$

$$b =$$

Include your plots below.