

Section 5.1 The Determinant Function

The determinant of a matrix is a way to map all of the entries of a matrix to a single number. This single number can be used to determine whether or not a square matrix is invertible or to give us a sense of how much error we might see in a numerical calculation involving matrices.

We build up the definition of the determinant by first defining the easiest ones to calculate.

If $A = [a_1] \in R$ is a 1×1 matrix then the $\det(A) = |A| = a_1$

$$\det([2]) = 2$$

If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is a 2×2 matrix then $\det(A) = |A| = ad - bc$

If you recall the inverse of a 2×2 matrix was

$$A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$\frac{1}{\det(A)}$

and it was exactly when $(ad - bc) = 0$ that the matrix was not invertible. In fact:

An $n \times n$ matrix A is invertible if and only if $\det(A) \neq 0$

If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$ then the determinant is given by,

$$\det(A) = |A| = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}$$

$(1)(5)(0) + 2(6)(0) + 3(4)(0) - (1)(6)(0) - 2(4)(0) - (3)(5)(0)$

Formulas for determinants of larger matrices can be written down, but they get increasingly messy. On the next page we will learn a procedural way to calculate a determinant which relies on the definitions for smaller matrices here.

Is the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 0 & 0 & 0 \end{bmatrix}$ invertible?

$$(1)(5)(0) + 2(6)(0) + 3(4)(0) - (1)(6)(0) - 2(4)(0) - (3)(5)(0) = 0$$

is A invertible NO!

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To be able to calculate Determinants of larger matrices we need to introduce the idea of a cofactor

Let A be an $n \times n$ matrix the cofactor C_{ij} of if the ij^{th} entry a_{ij} is

$$C_{ij} = (-1)^{i+j} \det(M_{ij})$$

where M_{ij} is the submatrix formed by omitting the entries in the i^{th} row and the j^{th} column of A

Example: What is the cofactor of the entry in the second row and first column in the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$C_{21} = (-1)^{2+1} \det \begin{bmatrix} 2 & 3 \\ 8 & 9 \end{bmatrix}$$

$$= -1 (2(9) - 8(3)) = -(18 - 24) = 6$$

What about the third row and third column?

$$i=3 \quad j=3$$

$$C_{33} = (-1)^{(3+3)} \det \begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix}$$

$$= (-1)^6 (1(5) - 4(2))$$

$$= 1(5 - 8) = -3$$

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Let A be the $n \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

Then the determinant of A is

$$\det(A) = a_{11}C_{11} + a_{12}C_{12} + \cdots + a_{1n}C_{1n}$$

Cofactor Definition for Reference

Let A be an $n \times n$ matrix the cofactor C_{ij} of if the ij^{th} entry a_{ij} is

$$C_{ij} = (-1)^{i+j} \det(M_{ij})$$

where M_{ij} is the submatrix formed by omitting the entries in the i^{th} row and the j^{th} column of A

Ex Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 6 \\ 7 & -1 & 1 \end{bmatrix}$
Find $\det(A)$

$$\rightarrow \begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

$$1C_{11} + 0C_{12} + 2C_{13}$$

$$(1) \begin{vmatrix} 1 & 6 \\ -1 & 1 \end{vmatrix} - 0 \begin{vmatrix} 2 & 6 \\ 7 & 1 \end{vmatrix} + 2 \begin{vmatrix} 2 & 1 \\ 7 & -1 \end{vmatrix}$$

$$(1)(1(1) - (-1)(6)) - 0(-) + 2(2(-1) - 7(1)) \\ = 7 + 0 + -18 = \boxed{-11}$$

In the calculation above it was nice that we had a 0 in the first row as it reduced the number of calculations.

Consider the matrix

$$A = \begin{bmatrix} 0 & 2 & 2 \\ 0 & 1 & 6 \\ 0 & -1 & 1 \end{bmatrix}$$

There are a lot of zeros in that first column! Using the definition above we would have expanded along the first row, however we can actually expand across any row or column.

$\det(A)$?

$$1 \begin{vmatrix} 1 & 6 \\ -1 & 1 \end{vmatrix} - 0 \begin{vmatrix} 2 & 2 \\ -1 & 1 \end{vmatrix} + 0 \begin{vmatrix} 2 & 2 \\ 1 & 6 \end{vmatrix}$$

$$1((1)(1) - (-1)(6)) = \boxed{7}$$

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Let A be an $n \times n$ matrix then

(a) $\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \dots + a_{in}C_{in}$ (Expansion across row i)

(b) $\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \dots + a_{nj}C_{nj}$ (Expansion across row j)

(a) and (b) are called the cofactor expansions

Example: Calculate the determinant of the matrix

$$A = \begin{vmatrix} -1 & 2 & 7 & -5 & 8 \\ 0 & 3 & 4 & 1 & -9 \\ 0 & 0 & -2 & 4 & 11 \\ 0 & 0 & 0 & -4 & 5 \\ 0 & 0 & 0 & 0 & 1 \end{vmatrix}$$

$$0 \begin{vmatrix} 2 & 7 & -5 & 8 \\ 3 & 4 & 1 & -9 \\ 0 & -2 & 4 & 11 \\ 0 & 0 & -4 & 5 \end{vmatrix} - 0 \begin{vmatrix} -1 & 7 & -5 & 8 \\ 0 & 4 & 1 & -9 \\ 0 & -2 & 4 & 11 \\ 0 & 0 & -4 & 5 \end{vmatrix} + 0 \begin{vmatrix} -1 & 2 & -5 & 8 \\ 0 & 3 & 1 & -9 \\ 0 & 0 & 4 & 11 \\ 0 & 0 & -4 & 5 \end{vmatrix} - 0 \begin{vmatrix} -1 & 2 & 7 & -5 \\ 0 & 3 & 4 & 1 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & -4 \end{vmatrix} + (1) \begin{vmatrix} -1 & 2 & 7 & -5 \\ 0 & 3 & 4 & 1 \\ 0 & 0 & -2 & 4 \\ 0 & 0 & 0 & -4 \end{vmatrix}$$

$$= 1 \left(-1 \begin{vmatrix} 3 & 4 & 1 \\ 0 & -2 & 4 \\ 0 & 0 & -4 \end{vmatrix} - 0 \begin{vmatrix} 2 & 7 & -5 \\ 0 & -2 & 4 \\ 0 & 0 & -4 \end{vmatrix} + 0 \begin{vmatrix} -1 & 7 & -5 \\ 0 & 4 & 1 \\ 0 & -2 & 4 \end{vmatrix} - 0 \begin{vmatrix} -1 & 2 & -5 \\ 0 & 3 & 1 \\ 0 & 0 & 4 \end{vmatrix} \right)$$

$$= -1 \left(3 \begin{vmatrix} -2 & 4 \\ 0 & -4 \end{vmatrix} - 0 \begin{vmatrix} 3 \\ 1 \end{vmatrix} + 0 \begin{vmatrix} 3 \\ 1 \end{vmatrix} \right) = -1 (3(-2(-4) + 0(4))) = -24$$

$$\left(\frac{1}{10} - \frac{1}{9} \right) \cdot$$

$$1 = \boxed{-24}$$

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If A is a triangular $n \times n$ matrix then $\det(A)$ is the product of the terms along the diagonal

Proof:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

$$a_{11} \begin{vmatrix} a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{vmatrix} + 0 -$$

$$a_{11} \left(a_{22} \begin{vmatrix} a_{33} & a_{34} & \cdots & a_{3n} \\ 0 & a_{44} & \cdots & a_{4n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & a_{nn} \end{vmatrix} \right) + 0 - \dots$$

$$a_{11} \left(a_{22} \left(a_{33} \begin{vmatrix} a_{44} & \cdots & a_{4n} \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nn} \end{vmatrix} \right) \right) + 0$$

$$a_{11} a_{22} a_{33} a_{44} \cdots a_{nn} = a_{11} a_{22} a_{33} \cdots a_{nn}$$



Let A be a square matrix then $\det(A) = \det(A^T)$

A formal proof requires induction, but this can be intuitively understood because of what transposition is. For transposition all we do is swap rows for columns, and since we can expand across any row or column these determinants should be the same. If we calculated $\det(A)$ by expanding along the first row then we could just calculate $\det(A^T)$ by expanding across the first row.

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If a square matrix A has a row or columns of zeros that is it's determinant?

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

+ - + ... +

$$\begin{bmatrix} a_{11} & 0 & a_{13} & \cdots & a_{1n} \\ a_{21} & 0 & a_{23} & \cdots & a_{2n} \\ a_{31} & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

$$\det(A) = 0 \quad | \quad -0 \quad | \quad +0 \quad | \quad +0 \quad +0 = 0$$

If A is a square matrix and if

- (a) A has a row or column of zeros $\det(A) = 0$
- (b) A has two identical rows or columns $\det(A) = 0$

(b) is easy to see for 2×2 matrices

$$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = ab - ab = 0$$

$$\begin{vmatrix} a & a \\ c & c \end{vmatrix} = ac - ca = 0$$

To prove it for all $n \times n$ we would assume it is true for all matrices of size $(n-1) \times (n-1)$ and then show it is true for $n \times n$. This is proof by induction.

→ The trick is expand across any row or column that is not an identical one. Then every submatrix you use to calculate cofactors would be of size $(n-1) \times (n-1)$ and will have an identical row or column, since you did not expand across those. Thuse their determinants are all zero and so all the cofactors are zero and so is the $\det(A)$. Then since it was obviously true for 2×2 it is true for 3×3 , since the sub matrices of a 3×3 are 2×2 . If it is true for 3×3 then it is true for 4×4 since the submatrices for a 4×4 are 3×3 and so on!

$$\begin{vmatrix} a & b & c & d \\ \# & \# & \# & \# \\ a & b & c & d \\ \# & \# & \# & \# \end{vmatrix} \rightarrow \# \begin{vmatrix} b & c & d \\ \# & \# & \# \end{vmatrix} + \# \begin{vmatrix} a & c & d \\ \# & \# & \# \end{vmatrix} + \dots$$

$$\# \left(\begin{vmatrix} \# & c & d \\ \# & c & d \end{vmatrix} - \begin{vmatrix} b & d \\ b & d \end{vmatrix} \right)$$

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If $A, B \in R^{n \times n}$ then

$$\det(AB) = \det(A) \det(B)$$

More on this next section.

octave:14> A=randi([-10 10],4,4)
A =

```
-6  -6  -7   5
-10  4   2   9
-8  10   8  10
-3   5  -6 -10
```

octave:13> B=randi([-10 10],4,4)
B =

```
-2  -1  -6  -9
 4  -2  -8   7
 4   8   6  -9
-10 -8   7  10
```

octave:15> det(A)

ans = -2402.0

octave:16> det(B)

ans = -120.00

octave:18> A*B

ans =

```
-90  -78   77  125
-46  -54  103  190
-12  -28   86  170
102   25 -128   16
```

octave:19> det(A*B)

ans = 2.8824e+05

octave:17> det(A)*det(B)
ans = 2.8824e+05

The Unifying Theorem Expanded Again!

Let $S = \{\vec{a}_1, \dots, \vec{a}_n\}$ be a set of n vectors in R^n , let $A = [\vec{a}_1 \ \vec{a}_2 \ \dots \ \vec{a}_n]$. Let $T: R^n \rightarrow R^n$ be given by $T(\vec{x}) = A\vec{x}$. Then the following are equivalent:

- (a) S spans R^n
- (b) S is linearly independent
- (c) $Ax = b$ has a unique solution for all $\vec{b} \in R^n$
- (d) T is onto
- (e) T is one-to-one
- (f) A is invertible
- (g) $\ker(T) = \{0\}$ or equivalently the Nullspace(A) = {0}
- (h) S is a basis for R^n
- (i) $\text{col}(A) = R^n$
- (j) $\text{row}(A) = R^n$
- (k) $\text{Rank}(A) = n$
- (l) $\det(A) \neq 0$