

Section 5.2 Properties of the Determinant

The determinant can tell us a lot, such as whether a system has a unique solution or not, or whether a set of vectors is linearly independent. As such it is nice to know what happens to it when we multiply by other matrices or perform row operations, or take an inverse.

$$\rightarrow (-1) \left(a_{i1}^A (-1)^{i+1} |M_{i1}^A| + \dots + a_{in}^A (-1)^{i+n} |M_{in}^A| \right) = -\det(A)$$

Let A be a square matrix

(a) Suppose that B is produced by interchanging rows of A . Then $\det(A) = -\det(B)$

Let's just compare the cofactor expansions across R_i of A and R_{i+1} of B

$$A \rightarrow a_{i1}^A (-1)^{i+1} |M_{i1}^A| + a_{i2}^A (-1)^{i+2} |M_{i2}^A| + \dots + a_{in}^A (-1)^{i+n} |M_{in}^A| = \det(A)$$

$$B \rightarrow a_{(i+1)1}^B (-1)^{i+1+1} |M_{(i+1)1}^B| + a_{(i+1)2}^B (-1)^{(i+1)+2} |M_{(i+1)2}^B| + \dots + a_{(i+1)n}^B (-1)^{(i+1)+n} |M_{(i+1)n}^B|$$

$$= a_{i1}^A (-1)^{i+1} |M_{i1}^A| + a_{i2}^A (-1)^{i+2} |M_{i2}^A| + \dots + a_{in}^A (-1)^{i+n} |M_{in}^A|$$

(b) Suppose that B is produced by multiplying a row of A by c . Then $\det(A) = \frac{1}{c} \det(B)$

Let's look at the cofactor expansion across R_i which was multiplied by c

$$\det(B) = c a_{i1}^A (-1)^{i+1} |M_{i1}^A| + c a_{i2}^A (-1)^{i+2} |M_{i2}^A| + \dots + c a_{in}^A (-1)^{i+n} |M_{in}^A|$$

$$= c \left(a_{i1}^A (-1)^{i+1} |M_{i1}^A| + \dots + a_{in}^A (-1)^{i+n} |M_{in}^A| \right)$$

$$\det(B) = c \det(A) \rightarrow \det(A) = \frac{1}{c} \det(B)$$

(c) Suppose that B is produced by adding a multiple of one row of A to another. Then $\det(A)$

Let's just compare the cofactor expansions across R_j of B which is equal to $cR_i + R_j$ of A remembering that the submatrices will be the same across both rows.

$$\det(B) = a_{j1}^A (-1)^{j+1} |M_{j1}^A| + \dots + a_{jn}^A (-1)^{j+n} |M_{jn}^A|$$

$$= (c a_{i1}^A + a_{j1}^A) (-1)^{j+1} |M_{j1}^A| + \dots + (c a_{in}^A + a_{jn}^A) (-1)^{j+1} |M_{jn}^A|$$

$$\cancel{c a_{i1}^A (-1)^{j+1} |M_{j1}^A| + \dots + c a_{in}^A (-1)^{j+n} |M_{jn}^A|} + a_{j1}^A (-1)^{j+1} |M_{j1}^A| + \dots + a_{jn}^A (-1)^{j+n} |M_{jn}^A|$$

$$= 0 + \det(A)$$

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Why is $a_{i1}C_{j1} + a_{i2}C_{j2} + \dots + a_{in}C_{jn} = 0$ if $j \neq i$ (1)st row / m_j

A 3×3 example

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

Let $i = 2, j = 1$ then our determinant is

"cofactor expansion"

$$a_{21} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{22} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{23} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$a_{21}(a_{22}a_{33} - a_{32}a_{23}) - a_{22}(a_{21}a_{33} - a_{31}a_{23}) + a_{23}(a_{21}a_{32} - a_{31}a_{22})$$

$$a_{21}a_{22}a_{33} - a_{21}a_{32}a_{23} - a_{22}a_{21}a_{33} + a_{22}a_{31}a_{23} + a_{23}a_{21}a_{32} - a_{23}a_{31}a_{22} = 0$$

i could have been anything except 1! (to get zero that is)

$$a_{i1}a_{22}a_{33} - a_{i1}a_{32}a_{23} - a_{i2}a_{21}a_{33} + a_{i2}a_{31}a_{23} + a_{i3}a_{21}a_{32} - a_{i3}a_{31}a_{22} = 0$$

In fact we could show this with general i and each $j \neq i$ to show all cases here.

This fact is general for all matrices! Kind of cool.

Let A be a square matrix

(a) Suppose that B is produced by interchanging rows of A . Then $\det(A) = -\det(B)$

(b) Suppose that B is produced by multiplying a row of A by c . Then $\det(A) = \frac{1}{c} \det(B)$

(c) Suppose that B is produced by adding a multiple of one row of A to another. Then $\det(A)$

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$$\det(I) = \begin{vmatrix} 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & \vdots & \ddots \end{vmatrix} = 1$$

First Let's Prove that $\det(AB) = \det(A) \det(B)$, to do so we need the following lemma

If both E, B are $n \times n$ matrices and E is an elementary matrix then $\det(EB) = \det(E) \det(B)$

Let E_1 be an elementary matrix for row swap.

$$\det(E_1) = -\det(I) = -1$$

$$\det(E_1 B) = -\det(B)$$

$$\det(E_1) \det(B) = (-1) \det(B)$$

Let E_2 be an elementary matrix for multiplication of a row.

$$\det(E_2) = \det(I^{cR_i}) = c \det(I) = c(1)$$

$$\det(E_2 B) = c \det(B)$$

$$\det(E_2) \det(B) = c \det(B)$$

Let E_3 be an elementary matrix for $cR_i + R_j$.

$$\det(E_3) = \det(I) = 1$$

$$\det(E_3 B) = \det(B)$$

$$\det(E_3) \det(B) = (1) \det(B)$$

$$\det(AB) = \det(A) \det(B)$$

Let $E_1, E_2, \dots, E_k$ be the elementary rows which take A to rref form that is $E_k \cdots E_2 E_1 A = \text{rref}(A)$

Case 1 $\det(A) = 0$ meaning A in rref form has a zero in a pivot position.

$$\begin{matrix} \# & \# & \# & \# \\ \# & \# & \# & \# \\ 0 & 0 & 0 & 0 \end{matrix}$$

$$\det(A) \det(B) = 0$$

$$\det(E_k \cdots E_1) \det(AB) = \det(E_k \cdots E_1 AB)$$

$$\begin{bmatrix} \# & \# & \# & \# \\ \# & \# & \# & \# \\ 0 & 0 & 0 & 0 \\ \# & \# & \# & \# \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}$$

$$\det(E_k \cdots E_1) \det(AB) = 0 = \det(\text{rref}(A) B) = 0$$

$$\Rightarrow \det(AB) = 0$$

Row of zeros

Case 2 $\det(A) \neq 0$ in which case A has a 1 in each pivot position, which means $E_k \cdots E_2 E_1 A = I$ or that

$$A = (E_k \cdots E_2 E_1)^{-1} I = (E_k \cdots E_2 E_1)^{-1} = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$$

$$\det(AB) = \det(E_1^{-1} E_2^{-1} \cdots E_k^{-1} B)$$

$$= \det(E_1^{-1}) \det(E_2^{-1}) \cdots \det(E_k^{-1}) \det(B)$$

$$= \det(E_1^{-1} E_2^{-1} \cdots E_k^{-1}) \det(B)$$

$$= \det(A) \det(B)$$

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If A is a square invertible matrix

$$\det(A^{-1}) = \frac{1}{\det(A)}$$

Proof

Consider $\det(AA^{-1}) = \det(A)\det(A^{-1})$

$$\hookrightarrow \det(I)$$

$$\downarrow$$

$$1 = \det(A)\det(A^{-1})$$

$$\frac{1}{\det(A)} = \det(A^{-1})$$

□

Let A be a square matrix, A is invertible if and only if $\det(A) \neq 0$

Suppose A has an inverse, then $\text{rref}(A) = I$ how do $\det(A)$ and $\det(I)$ compare?

$$B = \text{rref}(A)$$

$$B = \det(I)$$

$$\det(B) = c \pm \det(A)$$

$$\downarrow$$

$$\det(I) = 1 = c \pm \det(A)$$

$$\Rightarrow \det(A) = \frac{1}{\pm c} \neq 0$$

Suppose $\det(A) \neq 0$ and consider the matrix B , the echelon form of A , obtained only through Row swaps and additive row operations. What must the structure of B be?

$$\det\left(\begin{bmatrix} b_{11} & * & - \\ & b_{22} & - \\ 0 & & b_{nn} \end{bmatrix}\right) = \underline{\underline{b_{11}b_{22} \cdots b_{nn} \neq 0}}$$

$\Rightarrow A$ is invertible

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Find the determinant of the original matrix A which was transformed to B

$$\begin{aligned} & \rightarrow R_2 \leftrightarrow R_3 \quad (A) \\ & \rightarrow R_2 + R_3 \rightarrow R_3 \quad (B) \\ & \rightarrow -7R_2 + R_4 \rightarrow R_4 \quad (B') \\ & \rightarrow R_3 \leftrightarrow R_4 \quad (B'') \\ & \rightarrow -5R_4 \rightarrow R_4 \quad (B''') \end{aligned}$$

$$B = \begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 3 & 1 & 1 \\ 0 & 0 & 1 & 6 \\ 0 & 0 & 0 & 5 \end{bmatrix}$$

$$\det(B) = (1)(3)(1)(5) = 15$$

$$\det(B^{-5R_4}) = \det(B)$$

$$\downarrow$$

$$-5 \det(B') = 15$$

$$\det(B') = -3$$

$$\det(B'') = 3$$

$$\det(B''') = 3$$

$$\det(B^4) = 3$$

$$\det(A) = -3$$