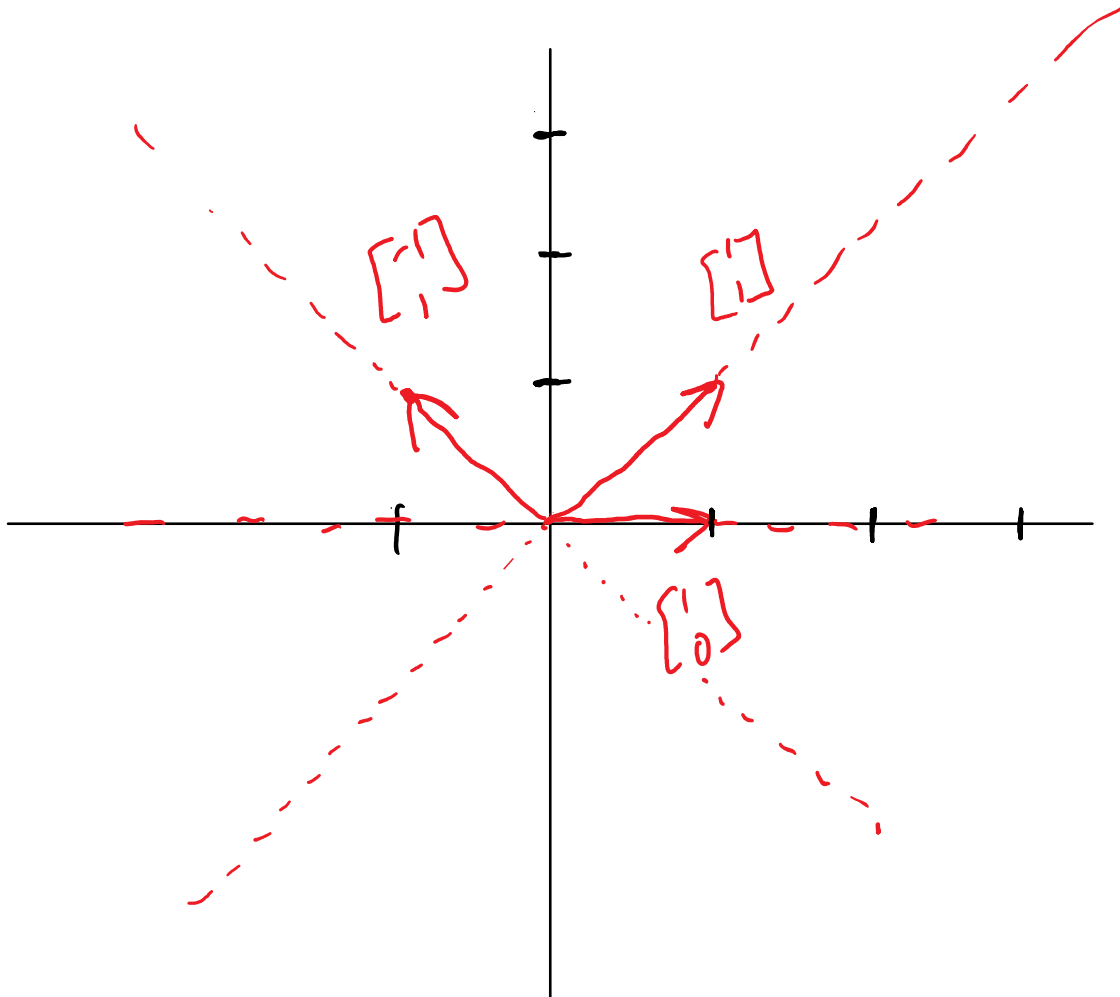


Section 6.1 Eigen Values and Eigen Vectors

Eigen is a German word meaning own. An eigen vector of a linear transformation is a special vector that intrinsically characterizes the linear transformation itself. In essence an Eigen vector is a vector for which its associated linear transformation A does not change under the action of A , except by a scaling.

Ex.

Let $A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$ What happens to the vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$?



Section 6.1 Eigen Values and Eigen Vectors

For $A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$ It turns out that $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ are **eigenvectors of A**

The vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ was scaled by 2 and the vector $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ was scaled by 1. These are called **eigen values**.

Let A be an $n \times n$ matrix. Then a nonzero vector $\vec{u}$ is an **eigenvector** of A if there exists a scalar λ such that

$$A\vec{u} = \lambda\vec{u}$$

The scalar λ is called an **eigenvalue** of A

How do we find eigen values and vectors?

If we rearrange the equation in the definition

$$A\vec{u} = \lambda\vec{u} \rightarrow A\vec{u} - \lambda\vec{u} = 0 \rightarrow A - \lambda I\vec{u} = 0 \rightarrow (A - \lambda I)\vec{u} = 0$$

$$\left(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & 1 \end{bmatrix} \right) \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = 0$$

$$\left(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} - \begin{bmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & 0 & \vdots \\ \vdots & 0 & \ddots & 0 \\ 0 & \cdots & 0 & \lambda \end{bmatrix} \right) \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = 0$$

$$\left(\begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{bmatrix} \right) \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = 0$$

Section 6.1 Eigen Values and Eigen Vectors

$$\left(\begin{bmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{bmatrix} \right) \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = 0$$

So in looking for eigen values and eigen vectors we are searching values λ and **non zero** vectors $\vec{u}$ such that when we subtract λ from the diagonal entries of A and multiply that matrix by $\vec{u}$ we get zero. We can think of $\vec{u}$ as a vector in the null space of $(A - \lambda I)$.

Let $S = \{\vec{a}_1, \dots, \vec{a}_n\}$ be a set of n vectors in R^n , let $A = [\vec{a}_1 \ \vec{a}_2 \ \cdots \ \vec{a}_n]$. Let $T: R^n \rightarrow R^n$ be given by $T(\vec{x}) = A\vec{x}$. Then the following are equivalent:

- (a) S spans R^n
- (b) S is linearly independent
- (c) $Ax = b$ has a unique solution for all $\vec{b} \in R^n$
- (d) T is onto
- (e) T is one-to-one
- (f) A is invertible
- (g) $\ker(T)=0$ or equivalently the Nullspace(A)=0**
- (h) S is a basis for R^n
- (i) $\text{col}(A) = R^n$
- (j) $\text{row}(A) = R^n$
- (k) $\text{Rank}(A) = n$
- (l) $\det(A) \neq 0$**

What we want here is for $\text{Nullspace}(A - \lambda I) \neq 0$ we want nonzero vectors in the null space, those are the eigen vectors. From the unifying theorem above what does that say the $\det(A - \lambda I)$ must be?

This condition can be thought of geometrically as finding values of λ for which the matrix $(A - \lambda I)$ takes area to zero. That is collapses the plane onto the eigen vectors.

Section 6.1 Eigen Values and Eigen Vectors

So we want $\det(A - \lambda I) = 0$ this equation will allow us to find eigen values, then after we have those we can find the nullspaces of each $(A - \lambda I)$ to get eigenvectors.

Let's find the values for our previous example $A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$

We call the polynomial we get from $\det(A - \lambda I)$ the characteristic polynomial

To find our eigen vectors we will find the null spaces of

$$(A - 2I) = \begin{bmatrix} 2-2 & 1 \\ 0 & 1-2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}$$

$$(A - 1I) = \begin{bmatrix} 2-1 & 1 \\ 0 & 1-1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

Section 6.1 Eigen Values and Eigen Vectors

Let's do a more complicated example together

Find the Eigen Values and Eigen Vectors of the matrix $A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$

Step 1. Find the characteristic polynomial and its zeros will be the eigen values.

Step 2. Find the null spaces of the matrices $(A - \lambda_i I)$ those are your eigen vectors, each associated with an eigen value λ_i

$$(A - 1I)\vec{x} = \begin{bmatrix} 0 & 2 & 4 \\ 0 & -3 & 1 \\ 0 & 1 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad (A - (-1)I)\vec{x} = \begin{bmatrix} 2 & 2 & 4 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0 \quad (A - (-3)I)\vec{x} = \begin{bmatrix} 4 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

Section 6.1 Eigen Values and Eigen Vectors

```
octave:11> u1=[1; 0; 0]
u1 =
```

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

```
octave:19> A*u1
ans =
```

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

```
octave:12> u2=[-3; 1; 1]
u2 =
```

$$\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$$

```
octave:20> A*u2
ans =
```

$$\begin{bmatrix} 3 \\ -1 \\ -1 \end{bmatrix}$$

```
octave:13> u3=[-1/2; 1; 1]
u3 =
```

$$\begin{bmatrix} -1/2 \\ 1 \\ 1 \end{bmatrix}$$

```
octave:21> A*u3
ans =
```

$$\begin{bmatrix} 3/2 \\ 3 \\ -3 \end{bmatrix}$$

It works!

So far we have seen examples when there are as many eigen values and vectors as column vectors, is that always how it is?

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix} \quad \det(A - \lambda I) = \det \left(\begin{bmatrix} 1-\lambda & 2 & 4 \\ 0 & -2-\lambda & 1 \\ 0 & 0 & -2-\lambda \end{bmatrix} \right) = (1-\lambda)(-2-\lambda)(-2-\lambda) = 0$$

$$\lambda_1 = 1, \lambda_2 = -2, \lambda_3 = -2$$

```
octave:29> (A-(1)*eye(3))
ans =
```

$$\begin{bmatrix} 0 & 2 & 4 \\ 0 & -3 & 1 \\ 0 & 0 & -3 \end{bmatrix}$$

```
octave:27> rref(A-1*eye(3))
ans =
```

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_1 &= t \\ x_2 &= 0 \\ x_3 &= 0 \end{aligned}$$

$$\vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} t, \lambda_1 = 1$$

```
octave:30> (A-(-2)*eye(3))
ans =
```

$$\begin{bmatrix} 3 & 2 & 4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

```
octave:28> rref(A-(-2)*eye(3))
ans =
```

$$\begin{bmatrix} 1 & 2/3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\begin{aligned} x_3 &= 0 \\ x_2 &= t \\ x_1 &= -\frac{2}{3}t \end{aligned}$$

$$\vec{u}_2 = \begin{bmatrix} -2/3 \\ 1 \\ 0 \end{bmatrix} t, \lambda_2 = -2$$

In this case we only get two eigen vectors! However repeated eigen values does not always preclude a loss of eigen vectors.

Section 6.1 Eigen Values and Eigen Vectors

```
octave:41> A=[1 -2 1; -1 0 1; -1 -2 3]
A =
```

$$\begin{bmatrix} 1 & -2 & 1 \\ -1 & 0 & 1 \\ -1 & -2 & 3 \end{bmatrix}$$

```
octave:42> eig(A)
ans =
```

$$\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}$$

```
octave:44> rref(A-(0)*eye(3))
ans =
```

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

```
octave:45> (A-(2)*eye(3))
ans =
```

$$\begin{bmatrix} -1 & -2 & 1 \\ -1 & -2 & 1 \\ -1 & -2 & 1 \end{bmatrix}$$

one free paramater

```
octave:45> (A-(2)*eye(3))
ans =
```

$$\begin{bmatrix} -1 & -2 & 1 \\ -1 & -2 & 1 \\ -1 & -2 & 1 \end{bmatrix}$$

```
octave:46> rref(A-(2)*eye(3))
ans =
```

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

2 free paramaters

Section 6.1 Eigen Values and Eigen Vectors

The set of eigen vectors associated with an eigen value λ together with the zero vector forms a subspace.

Proof:

Let A be a square matrix with eigenvalue λ . The subspace of all eigenvectors associated with λ , together with the zero vector is called the eigenspace of λ .

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 0 & -2 & 1 \\ 0 & 0 & -2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & -2 & 1 \\ -1 & 0 & 1 \\ -1 & -2 & 3 \end{bmatrix}$$

Eigen space $\lambda_1 = 1$

$$\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\}$$

Eigen space $\lambda_1 = 0$

$$\text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

Eigen space $\lambda_2 = -2$

$$\text{span} \left\{ \begin{bmatrix} -\frac{2}{3} \\ 1 \\ 0 \end{bmatrix} \right\}$$

Eigen space $\lambda_2 = 2$

$$\text{span} \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Section 6.1 Eigen Values and Eigen Vectors

In the previous two examples we had a repeated eigen value , 2 and -2,. Each showed up twice in the characteristic polynomial and so we would say it has multiplicity 2. One had eigenspace with dimension 1 and one with dimension 2.

Let λ be an eigen value of an $n \times n$ matrix A of multiplicity m . Then the dimension of the eigen space is always less than or equal to m .

The Unifying Theorem Expanded Again!

Let $S = \{\vec{a}_1, \dots, \vec{a}_n\}$ be a set of n vectors in R^n , let $A = [\vec{a}_1 \ \vec{a}_2 \ \dots \ \vec{a}_n]$. Let $T: R^n \rightarrow R^n$ be given by $T(\vec{x}) = A\vec{x}$. Then the following are equivalent:

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- (g) $\ker(T)=0$ or equivalently the Nullspace(A)=0
- (h) S is a basis for R^n
- (i) $\text{col}(A) = R^n$
- (j) $\text{row}(A) = R^n$
- (k) $\text{Rank}(A) = n$
- (l) $\det(A) \neq 0$
- (m) $\lambda = 0$ is not an eigen vector of A**

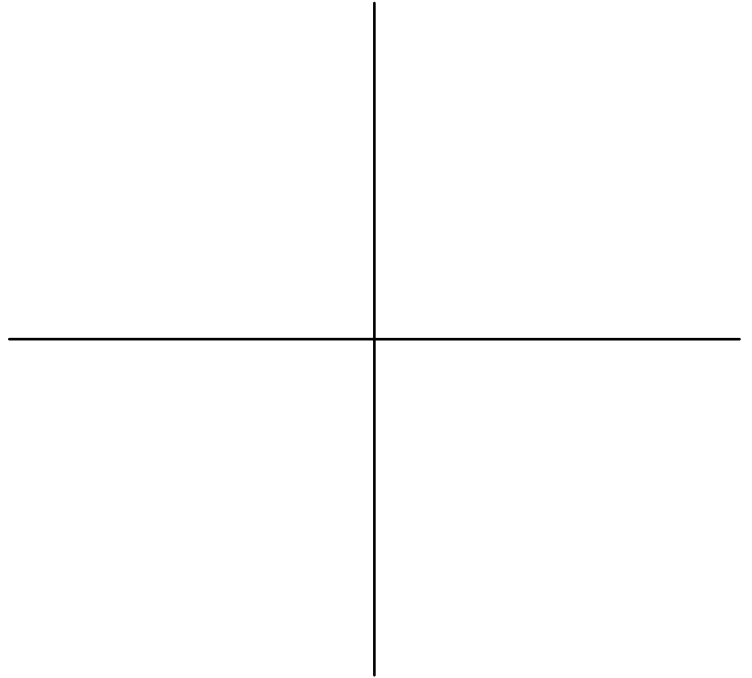
Section 6.1 Eigen Values and Eigen Vectors

When there are no eigen vectors!

Think about a rotation in 2d, every vector changes orientation!

$$\begin{bmatrix} \cos\left(\frac{-\pi}{2}\right) & -\sin\left(\frac{-\pi}{2}\right) \\ \sin\left(\frac{-\pi}{2}\right) & \cos\left(\frac{-\pi}{2}\right) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\det\left(\begin{bmatrix} 0-\lambda & 1 \\ -1 & 0-\lambda \end{bmatrix}\right) = 0 \rightarrow \lambda^2 + 1 = 0, \lambda = \pm i$$



In general we can have complex eigen values and complex eigenvectors, but not in the space of real numbers that we are sticking to in this class. However, complex eigenvalues of matrices say a lot about physical systems. If you take a differential equations class you will run into them there!

For the system of differential equations given by

$$\begin{bmatrix} \frac{dx_1}{dt} \\ \frac{dx_2}{dt} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

