

Section 6.2 Diagonalization

An $n \times n$ matrix A is diagonalizable if there exist $n \times n$ matrices P, D where D is diagonal and P invertible such that,

$$A = PDP^{-1}$$

How can we find P and D ?

Suppose A has n Linearly independent eigen vectors $\{\vec{u}_1 \dots \vec{u}_n\}$ with corresponding eigen values $\{\lambda_1, \dots, \lambda_n\}$

Let $P = [\vec{u}_1 \ \vec{u}_2 \ \dots \ \vec{u}_n]$ and $D_{ij} = [\lambda_i \delta_{ij}] = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \lambda_n \end{bmatrix}$

$D_{11} = \lambda_1 \delta_{11} = \lambda_1$
 $D_{12} = \lambda_1 \delta_{12} = 0$
 $D_{22} = \lambda_2 \delta_{22} = \lambda_2$
 $\delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$ (Kronecker delta)

Consider $AP = A[\vec{u}_1 \ \vec{u}_2 \ \dots \ \vec{u}_n] = [A\vec{u}_1 \ A\vec{u}_2 \ \dots \ A\vec{u}_n] = [\lambda_1 \vec{u}_1 \ \lambda_2 \vec{u}_2 \ \dots \ \lambda_n \vec{u}_n]$

Consider $PD = [\vec{u}_1 \ \vec{u}_2 \ \dots \ \vec{u}_n] \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & 0 & \lambda_n \end{bmatrix} = \begin{bmatrix} \lambda_1 u_{11} & \lambda_1 u_{12} & \dots & \lambda_1 u_{1n} \\ \lambda_2 u_{21} & \lambda_2 u_{22} & \dots & \lambda_2 u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_n u_{n1} & \lambda_n u_{n2} & \dots & \lambda_n u_{nn} \end{bmatrix}$

$$= \begin{bmatrix} \lambda_1 u_{11} + 0u_{12} + \dots + 0u_{1n} & 0u_{11} + \lambda_2 u_{12} + \dots + 0u_{1n} & \dots & 0u_{11} + 0u_{12} + \dots + \lambda_n u_{1n} \\ \lambda_1 u_{21} + 0u_{22} + \dots + 0u_{2n} & 0u_{21} + \lambda_2 u_{22} + \dots + 0u_{2n} & \dots & 0u_{21} + 0u_{12} + \dots + \lambda_n u_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 u_{n1} + 0u_{n2} + \dots + 0u_{nn} & 0u_{n1} + \lambda_2 u_{n2} + \dots + 0u_{nn} & \dots & 0u_{n1} + 0u_{n2} + \dots + \lambda_n u_{nn} \end{bmatrix}$$

$$= [\lambda_1 \vec{u}_1 \ \lambda_2 \vec{u}_2 \ \dots \ \lambda_n \vec{u}_n]$$

So, $AP = DP$ and since P is made up of linearly independent vectors, by assumption, it is invertible so..

$$AP = DP$$

$$AP = DP$$

$$APP^{-1} = PDP^{-1}$$

$$AI = PDP^{-1} \rightarrow A = PDP^{-1}$$

Section 6.2 Diagonalization

Diagonalize A if possible. (Find P and D such that $A = PDP^{-1}$ for the given matrix A . Enter your answer as one augmented matrix. If the matrix is not able to be diagonalized, enter DNE in any cell.)

$$\begin{bmatrix} -1 & 10 \\ -2 & 8 \end{bmatrix}$$

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$$[P|D] = \begin{bmatrix} \boxed{} & \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} & \boxed{} \end{bmatrix}$$

P is made of e vec

D is made of e vals

Find the eigen values and eigen vectors!

$$A = \begin{bmatrix} -1 & 10 \\ -2 & 8 \end{bmatrix}$$

$$\det(A - \lambda I) = 0$$

$$|(A - \lambda I)| = \begin{vmatrix} -1-\lambda & 10 \\ -2 & 8-\lambda \end{vmatrix} = (-1-\lambda)(8-\lambda) - (-2)(10) = 0$$

$$= \lambda^2 - 8\lambda + \lambda - 8 + 20$$

$$= \lambda^2 - 7\lambda + 12 = 0$$

$$(\lambda - 4)(\lambda - 3) = 0$$

$$\lambda_1 = 4 \quad \lambda_2 = 3$$

$$D = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$$

$$\lambda_1 = 4$$

$$\text{Null} \left(\begin{bmatrix} -1-4 & 10 \\ -2 & 8-4 \end{bmatrix} \right) = \text{Null} \left(\begin{bmatrix} -5 & 10 \\ -2 & 4 \end{bmatrix} \right)$$

$$\left[\begin{array}{cc|c} -5 & 10 & 0 \\ -2 & 4 & 0 \end{array} \right] \xrightarrow{-2R_1 + 5R_2} \left[\begin{array}{cc|c} -5 & 10 & 0 \\ 0 & 0 & 0 \end{array} \right]$$

$$\begin{array}{rcl} 10 & -20 & 0 \\ + & -10 & 20 & 0 \\ \hline 0 & 0 & 0 \end{array}$$

$$\left[\begin{array}{cc|c} -5 & 10 & 0 \\ 0 & 0 & 0 \end{array} \right] \xrightarrow{x_2 = t} \begin{array}{l} -5x_1 + 10x_2 = 0 \\ x_1 = \frac{-10t}{-5} = x_1 = 2t \end{array}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} t$$

$$P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$$

Section 6.2 Diagonalization

Diagonalize A if possible. (Find P and D such that $A = PDP^{-1}$ for the given matrix A . Enter your answer as one augmented matrix. If the matrix is not able to be diagonalized, enter DNE in any cell.)

$$\begin{bmatrix} -1 & 10 \\ -2 & 8 \end{bmatrix}$$

6a

$$[P \ D] = \begin{bmatrix} \boxed{} & \boxed{} & \boxed{} & \boxed{} \\ \boxed{} & \boxed{} & \boxed{} & \boxed{} \end{bmatrix}$$

P is made of e vec
 D is made of e vals

Find the eigen values and eigen vectors!

$$\text{Eigen values: } \det(A - \lambda I) = \begin{vmatrix} -1-\lambda & 10 \\ -2 & 8-\lambda \end{vmatrix} = (-1-\lambda)(8-\lambda) + 20 = \lambda^2 - 7\lambda - 8 + 20 = \lambda^2 - 7\lambda + 12$$

$$\lambda^2 - 7\lambda + 12 = 0$$

$$(\lambda - 4)(\lambda - 3) = 0$$

$$\lambda_1 = 4 \quad \lambda_2 = 3$$

Eigen vectors, find the null space of $(A - \lambda_i I)$

$$(A - 4I) = \begin{bmatrix} -1-4 & 10 \\ -2 & 8-4 \end{bmatrix} = \begin{bmatrix} -5 & 10 \\ -2 & 4 \end{bmatrix}$$

octave:11> A4=[-5 10; -2 4]
A4 =

$$\begin{bmatrix} -5 & 10 \\ -2 & 4 \end{bmatrix} \left| \begin{matrix} 0 \\ 0 \end{matrix} \right.$$

octave:13> rref(A4)
ans =

$$\begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix} \left| \begin{matrix} 0 \\ 0 \end{matrix} \right.$$

$$\begin{aligned} x_1 - 2x_2 &= 0 \\ 0x_1 + 0x_2 &= 0 \end{aligned}$$

$$x_2 = t \quad x_1 = 2t$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2t \\ t \end{bmatrix} = t \begin{bmatrix} 2 \\ 1 \end{bmatrix} \rightarrow \vec{u}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$(A - 3I) = \begin{bmatrix} -1-3 & 10 \\ -2 & 8-3 \end{bmatrix} = \begin{bmatrix} -4 & 10 \\ -2 & 5 \end{bmatrix}$$

octave:12> A3=[-4 10; -2 5]
A3 =

$$\begin{bmatrix} -4 & 10 \\ -2 & 5 \end{bmatrix} \left| \begin{matrix} 0 \\ 0 \end{matrix} \right.$$

octave:16> rref(A3)
ans =

$$\begin{bmatrix} 1 & -5/2 \\ 0 & 0 \end{bmatrix} \left| \begin{matrix} 0 \\ 0 \end{matrix} \right.$$

$$x_1 - \frac{5}{2}x_2 = 0$$

$$0x_1 + 0x_2 = 0$$

$$x_2 = t \quad x_1 = \frac{5}{2}t$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{5}{2}t \\ t \end{bmatrix} = t \begin{bmatrix} \frac{5}{2} \\ 1 \end{bmatrix} \rightarrow \vec{u}_2 = \begin{bmatrix} \frac{5}{2} \\ 1 \end{bmatrix}$$

$$\text{So, } P = \begin{bmatrix} 2 & 5/2 \\ 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$$

$\vec{u}_1$ $\vec{u}_2$

octave:17> P=[2 5/2; 1 1]
P =

$$\begin{bmatrix} 2 & 5/2 \\ 1 & 1 \end{bmatrix}$$

octave:18> D=[4 0; 0 3]
D =

$$\begin{bmatrix} 4 & 0 \\ 0 & 3 \end{bmatrix}$$

octave:20> P*D*inv(P)
ans =

$$\begin{bmatrix} -1 & 10 \\ -2 & 8 \end{bmatrix}$$

Section 6.2 Diagonalization

Matrix Powers

Suppose that A is diagonalizable, with $A = PDP^{-1}$. Then,

$$A^2 = (PDP^{-1})(PDP^{-1}) = PD(\cancel{P^{-1}P})DP^{-1} = PD^2P^{-1}$$

$$A^3 = A(A^2) = (PDP^{-1})(PD^2P^{-1}) = PD(\cancel{P^{-1}P})D^2P^{-1} = PD^3P^{-1}$$

And by induction

$$A^k = PD^kP^{-1}$$

Proof:

Suppose $A^k = PD^kP^{-1}$ consider $A^{k+1} = A(A^k)$

$$A(A^k) = A(PD^kP^{-1}) = (PDP^{-1})(PD^kP^{-1})$$

$$= PD(\cancel{P^{-1}P})D^kP^{-1} = PD^kD^kP^{-1} = PD^{k+1}P^{-1}$$

If true for $A^k \Rightarrow$ true for A^{k+1}

Since we have shown that $A^2 = PD^2P^{-1}$ by the argument above this implies that it is true for $A^{2+1} = A^3$. If it is true for A^3 by the same general argument above it is also true for $A^{3+1} = A^4$ and so on.

Ex Find a formula for A^k if $A = PDP^{-1}$ with

$$P = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \text{ and } D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$P^{-1} = \frac{1}{5(1) - 2(2)} \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix} = \frac{1}{1} \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$$

$$A^k = PD^kP^{-1}$$

$$D^k = \begin{bmatrix} 1^k & 0 \\ 0 & (-1)^k \end{bmatrix}$$

$$A^k = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (-1)^k \end{bmatrix} \begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 1^k(5) & 1^k(-2) \\ (-1)^k(-2) & (-1)^k(1) \end{bmatrix}$$

$$= \begin{bmatrix} 1(5) + (-1)^k(-2) & -2(1) + (-2)(-2) \\ 2(5) + 5(-1)^k(-2) & 2(-2) + 5(-1)^k(1) \end{bmatrix}$$

Section 6.2 Diagonalization

An $n \times n$ matrix A is diagonalizable if and only if its eigen vectors form a basis for R^n .

Proof:

Assume the eigen vectors of A form a basis for R^n , then as we showed on page one we can diagonalize A where $A = PDP^{-1}$ using $P = [\vec{u}_1, \dots, \vec{u}_n]$ and $D = [\lambda_i \delta_{ij}]$. With $\vec{u}_i$ the i th eigen vector of A and λ_i its eigen value.

See Pg 1

Now Assume A is diagonalizable.....

This means $A = PDP^{-1}$ with P being invertible. $\uparrow$

Since P is invertible its column vectors must be linearly independent by the unifying theorem. We just need to show that they are eigen vectors of A ! To do this consider $AP = PD$.

$$AP = PD(P^{-1}P)$$

$$AP = PD$$

$$A[\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n] = [\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n] \begin{bmatrix} d_{11} & & & \\ & d_{22} & & \\ & & \ddots & \\ & & & d_{nn} \end{bmatrix}$$

$$[A\vec{u}_1, A\vec{u}_2, \dots, A\vec{u}_n] = [\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n] \begin{bmatrix} d_{11} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \quad [\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n] \begin{bmatrix} 0 \\ d_{22} \\ 0 \\ \vdots \\ 0 \end{bmatrix} \dots$$

$$[A\vec{u}_1, A\vec{u}_2, \dots, A\vec{u}_n] = [d_{11}\vec{u}_1, d_{22}\vec{u}_2, \dots, d_{nn}\vec{u}_n]$$

$$\underbrace{A\vec{u}_1 = d_{11}\vec{u}_1}_{\vec{u}_1 \quad \lambda_1 = d_{11}}, \quad \underbrace{A\vec{u}_2 = d_{22}\vec{u}_2}_{\vec{u}_2 \quad \lambda_2 = d_{22}}, \quad \dots, \quad \underbrace{A\vec{u}_n = d_{nn}\vec{u}_n}_{\vec{u}_n \quad \lambda_n = d_{nn}}$$

Section 6.2 Diagonalization

If $\{\lambda_1, \dots, \lambda_k\}$ are distinct eigen values of a matrix A . Then any set of associated eigen vectors $\{\vec{u}_1, \dots, \vec{u}_k\}$ is linearly independent.

Proof: Assume they aren't and show they must be.

Suppose the set $\{\vec{u}_1, \dots, \vec{u}_k\}$ is linearly dependent. This means at least one of the vectors in the set is a linear combination of the others. With out loss of generality choose the dependent vector to $\vec{u}_i$ with the others all linearly independent, then we could say:

$$A \vec{u}_i = c_1 \vec{u}_1 + \dots + c_{i-1} \vec{u}_{i-1} + c_{i+1} \vec{u}_{i+1} + \dots + c_k \vec{u}_k \quad \leftarrow \quad c_1 \ c_2 \ \dots \ c_k$$

Since these are eigen vectors we can look at.

$$\begin{aligned} A \vec{u}_i &= c_1 A \vec{u}_1 + \dots + c_{i-1} A \vec{u}_{i-1} + c_{i+1} A \vec{u}_{i+1} + \dots + c_k A \vec{u}_k \text{ or...} \\ \lambda_i \vec{u}_i &= c_1 \lambda_1 \vec{u}_1 + \dots + c_{i-1} \lambda_{i-1} \vec{u}_{i-1} + c_{i+1} \lambda_{i+1} \vec{u}_{i+1} + \dots + c_k \lambda_k \vec{u}_k \text{ if } \lambda_i \neq 0 \text{ this means...} \\ \vec{u}_i &= \frac{c_1 \lambda_1}{\lambda_i} \vec{u}_1 + \dots + \frac{c_{i-1} \lambda_{i-1}}{\lambda_i} \vec{u}_{i-1} + \frac{c_{i+1} \lambda_{i+1}}{\lambda_i} \vec{u}_{i+1} + \dots + \frac{c_k \lambda_k}{\lambda_i} \vec{u}_k \end{aligned}$$

So this means we have a second valid, but different scalars, linear combination of the same vectors which give us $\vec{u}_i$ but we proved that every linear combination of the same set of vectors is unique so it must be instead that $\vec{u}_i$ was not in the span of its fellow eigen vectors. They must have all been linearly independent all along.

What if $\lambda_i = 0$? Well since all the eigen values are distinct it is the only one. Then we would have:

$\lambda_i \vec{u}_i = 0(\vec{u}_i) = 0 = c_1 \lambda_1 \vec{u}_1 + \dots + c_{i-1} \lambda_{i-1} \vec{u}_{i-1} + c_{i+1} \lambda_{i+1} \vec{u}_{i+1} + \dots + c_k \lambda_k \vec{u}_k$ and at least some of the c_i 's would be non zero by the assumption $\vec{u}_i$ was a linear combination of the other eigen vectors, further none of the other eigen values could be zero because they are all distinct, this however would mean that the linearly independence of the other vectors is violated and we reach another contradiction. It must just be that all of the eigen vectors are linearly independent.

Section 6.2 Diagonalization

Suppose that an $n \times n$ matrix A has only real eigen values. Then A is diagonalizable if and only if the dimension of each eigen space is equal to the multiplicity of the corresponding eigenvalue.

Suppose the dimension of the eigenspace of each eigen vector is equal to the multiplicity of the eigen value.

This means there are n eigen vectors that are linearly independent.

This implies that they form a basis for $\mathbb{R}^n$ and so by what we proved earlier

An $n \times n$ matrix A is diagonalizable if and only if its eigen vectors form a basis for $\mathbb{R}^n$.

A is diagonalizable.

Now suppose that A is diagonalizable. This means there must be n linearly independent eigen vectors. If we keep in mind what we showed in section 6.1.,

Let λ be an eigen value of an $n \times n$ matrix A of multiplicity m . Then the dimension of the eigen space is always less than or equal to m .

We have two cases to consider....

Case 1: All of the eigen values have multiplicity 1

This means the dimension of the eigen space of each eigen value is exactly 1, the multiplicity of the eigen value. Since each eigen value must have an eigen vector they cannot have dimension 0.

Case 2: Some of the eigen values have multiplicity greater than 1. Let one of them λ have multiplicity m . Suppose $\dim(\text{eigenspace}(\lambda)) < m$. Then we have less than n eigen vectors which means that we don't have enough vectors to form a basis for $\mathbb{R}^n$, yet A was diagonalizable this whole time, so our assumption was wrong, the $\dim(\text{eigenspace}(\lambda)) = m$ so that we don't violate the other two theorems above and we have our result.

If A is an $n \times n$ matrix with n distinct eigen values, then A is diagonalizable.

Proof: Any Ideas? (Think about the theorem we just proved above)