

# Section 7.1 Vector Spaces

We have been playing around in one two particular vector spaces, one is  $R^n$  the space of n-dimensional vectors, the other is  $R^{n \times n}$  the space of  $n \times n$  matrices. However, a vector space is a much more general concept and one that is **extremely useful** in mathematics and applied mathematics.

What makes a vector space.

A vector space consists of a set  $V$  of objects we refer to as vectors together with operations of addition and scalar multiplication on the vectors that satisfy each of the following.

- (1) If  $\vec{v}_1, \vec{v}_2 \in V$  then so is  $\vec{v}_1 + \vec{v}_2$  (closure under addition)
- (2) If  $c \in R$  is a scalar and  $\vec{v} \in V$  then so is  $c \times \vec{v}$  (closure under scalar multiplication)
- (3) There is a zero vector in  $V$ ,  $\vec{0}$ , such that  $\vec{0} + \vec{v} = \vec{v}$  for all  $\vec{v} \in V$  (an additive identity)
- (4) For every  $\vec{v} \in V$  there is an  $\vec{v}^* \in V$  such that  $\vec{v}^* + \vec{v} = \vec{0}$ . (an additive inverse)  
**(note we denote  $\vec{v}^*$  as  $-\vec{v}$  hence forth)**
- (5) For all  $\vec{v}_1, \vec{v}_2, \vec{v}_3 \in V$  and scalars  $c_1, c_2 \in R$  the following properties hold:
  - (a)  $\vec{v}_1 + \vec{v}_2 = \vec{v}_2 + \vec{v}_1$  (additive commutivity)
  - (b)  $(\vec{v}_1 + \vec{v}_2) + \vec{v}_3 = \vec{v}_1 + (\vec{v}_2 + \vec{v}_3)$  (additive associativity)
  - (c)  $c_1(\vec{v}_1 + \vec{v}_2) = c_1\vec{v}_1 + c_1\vec{v}_2$  (scalars distribute across vectors)
  - (d)  $(c_1 + c_2)\vec{v}_1 = c_1\vec{v}_1 + c_2\vec{v}_1$  (vectors distribute across scalars)
  - (e)  $(c_1c_2)\vec{v}_1 = c_1(c_2\vec{v}_1)$  (associative scalar multiplication)
  - (f)  $1 \cdot \vec{v}_1 = \vec{v}_1$  (scalar multiplication identity)

Let's prove that the set of polynomials of degree  $n$  is a vector space.

That is the set  $P = \{f(x): R \rightarrow R \mid f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0, a_i \in R\}$

Let  $q(x), p(x) \in P$ . Is  $p(x) + q(x) \in P$ ?

Let  $p(x) \in P$  is  $c_1 p(x) \in P$ ?

Is there a zero vector?

Is there an additive inverse?

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The functions in our space are just made of real numbers and all of these are true for the real numbers!

Let's look at an example of another vector space made of functions.

$$\text{Let } V = \{f(x): R \rightarrow R \mid f(x) \text{ is continuous on all of } R\}$$

Some examples of vectors from this space,  $f(x) = x^2 + 1$ ,  $g(x) = \sin(x)$ ,  $h(x) = e^x$ .

We can see this is a vector space pretty quickly by noting that the sum of two continuous functions is continuous (closure under addition), constant multiples of continuous functions are (closure under scalar multiplication),  $f(x) + 0 = 0 + f(x) = f(x)$  (additive identity),  $f(x) + (-f(x)) = 0$  (additive inverse), and (5) again because we are going from  $R \rightarrow R$ .

This is the cool part, if this is a vector space we should be able to take linear combinations of functions and get other, specific functions! You have seen this in calculus with Taylor series!

$$\sin(x) = x - \frac{1}{6}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \frac{1}{9!}x^9 + \dots$$

$$\cos(x) = 1 + \frac{1}{2}x^2 + \frac{1}{4!}x^4 + \frac{1}{6!}x^6 + \dots$$

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots$$

So here we are taking linear combinations of  $\{1, x, x^2, x^3, \dots\}$  and obtaining other functions in the vector space. We might be tempted to think of this as a basis, and it is in a sense, but that is out of the scope of our course. We will be focused on finite bases.

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Example of set which is not a vector space.

The set  $P = \{f(x): R \rightarrow R \mid f(x) = ax^2 + b \text{ with } a, b \in Q\}$   $Q = \text{the set of rational numbers.}$

*closure under addition*

Which one of these does not apply?

*closure under scalar multiplication*

*existence of an additive inverse*

Subspaces in abstract vector spaces have to satisfy the same requirements as in  $R^n$

A subset of  $S \subseteq V$ , with  $V$  being a vector space, is a **subspace** if  $S$  satisfies the following three conditions

(a)  $S$  contains  $\vec{0}$

(b) if  $\vec{u}$  and  $\vec{v}$  are in  $S$ , the  $\vec{u} + \vec{v}$  is also in  $S$ . (closure under addition)

(c) If  $r$  is a real number and  $U$  is in  $S$ , then  $r\vec{u}$  is also in  $S$ . (closure under scalar multiplication)

Let  $V = \{f(x): R \rightarrow R \mid f(x) \text{ is continuous on all of } R\}$  and  $S \subseteq V$  be the set of all  $f(x) \in V$  that satisfy

$$\frac{d^2f}{dx^2} + f(x) = 0$$

is a subspace of  $V$

*Closure under addition*

*Closure under scalar multiplication*

*zero vector*

*existence of additive inverse*

# Section 7.1 Vector Spaces

## Examples of Vector Spaces (THIS LIST IS FAR FROM EXHAUSTIVE)

Euclidean space,  $R^n$  with  $n > 0$  and an integer, together with component wise vector addition and scalar multiplication

Things like:  $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \text{ with } x_i \in R$

$P^n$  the set of degree  $n$  polynomials, together with the usual addition and scalar multiplication of polynomials.

Things like:  $\vec{x} = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$

$R^{n \times m}$ , the space of  $n \times m$  matrices together with componentwise matrix addition and scalar multiplication.

Things like:  $\vec{x} = \begin{bmatrix} x_{11} & x_{12} & \dots & x_{1m} \\ x_{21} & x_{22} & \dots & x_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nm} \end{bmatrix}$

$C[a, b]$ , the space of continuous functions on the interval  $[a, b]$  together with the usual definition of pointwise addition and scalar multiplication of functions.

Things like  $\vec{x} = f(x) = 2x + 1, f(x) = |x|$

$C(R)$ , The space of continuous functions on all of  $R$  together with the usual definition of pointwise addition and scalar multiplication of functions.

Things like  $\vec{x} = f(x) = 2x + 1, f(x) = |x|$

$T(m, n)$ , the set of linear transformations from  $R^m \rightarrow R^n$ , together with the usual definition of pointwise addition and scalar multiplication of functions.

Things like:  $\vec{x} = f(x, y, z, w) = \begin{bmatrix} x + y \\ z - w \end{bmatrix}$  or  $f(x, y, z, w) = \begin{bmatrix} x \\ 2y + z \\ w \\ 3w \\ x + z \end{bmatrix}$

The set of  $C^n[a, b]$  the set of  $n$  times differentiable functions on the interval  $[a, b]$  together with the usual definition of pointwise addition and scalar multiplication of functions.

Things like:  $\vec{x} = f(x) = \sin(x)$ , or  $f(x) = x^n$

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Let  $V$  be a vector space and suppose that  $\vec{v} \in V$ . Then,

- (a)  $\vec{0}$  is a vector in  $V$  then  $\vec{v} + \vec{0} = \vec{v}$
- (b) if  $-\vec{v}$  is the additive inverse of  $\vec{v}$  then  $\vec{v} + (-\vec{v}) = \vec{0}$
- (c) the additive inverse of  $\vec{v}$  is unique
- (d)  $0 \cdot \vec{v} = \vec{0}$
- (e)  $-1 \cdot \vec{v} = -\vec{v}$

Proof of (a) all we have from the definition of vector spaces is that  $\vec{0} + \vec{v} = \vec{v}$  but we can use this along with the idea of a commutativity in addition to prove (a)

From the definition of a vector space,  $\vec{0} + \vec{v} = \vec{v}$ . We also have property 5(a) which says we can commute addition, so from this we can conclude  $\vec{0} + \vec{v} = \vec{v}$ . This may seem silly, but often in mathematics we like to ask, "What is the least we can assume to get the properties we want?". The more general we can be the more mathematical objects we can stuff into our framework.

Proof of (b) This is also true by the assumption of commutativity!

Proof of (c). This one is more fun, Let's prove it by assuming it is not!

Let  $\vec{v} \in V$  and assume there are  $(-\vec{v}_1), (-\vec{v}_2) \in V$  such that  $\vec{v} + (-\vec{v}_1) = \vec{0}$  and  $\vec{v} + (-\vec{v}_2) = \vec{0}$

$$\vec{v} + (-\vec{v}_1) = \vec{v} + (-\vec{v}_2)$$

Add  $\vec{0}$  to both sides

$$(-\vec{v}_1) + \vec{v} + (-\vec{v}_1) = (-\vec{v}_1) + \vec{v} + (-\vec{v}_2)$$

It is important to note, we have not assumed we can "add to both sides" here it is just that  $\vec{v} + (-\vec{v}_1) = \vec{0}$  and  $\vec{v} + (-\vec{v}_2) = \vec{0}$  so the above reads as  $(-\vec{v}_1) + \vec{0} = (-\vec{v}_1) + \vec{0}$  which is just  $-\vec{v}_1 = -\vec{v}_1$ .

Use the associative law

$$((-\vec{v}_1) + \vec{v}) + (-\vec{v}_1) = ((-\vec{v}_1) + \vec{v}) + (-\vec{v}_2)$$

And by property (a) above  $(-\vec{v}_1) + \vec{v} = \vec{v} + (-\vec{v}_1) = \vec{0}$  so we just have

$$(\vec{0}) + (-\vec{v}_1) = (\vec{0}) + (-\vec{v}_2)$$

Or

$$(-\vec{v}_1) = (-\vec{v}_2)$$

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Proof of (d)

By existence of additive identity

By existence of additive inverse

By the associative property for vector addition

By the scalar multiplicative identity  $1\vec{v} = \vec{v}$

By distributive property for scalar multiplication

By the additive identity ( $R$  is also a vector space!)

By the scalar multiplicative identity  $1\vec{v} = \vec{v}$

By the existence of additive identity

Proof of (d) it is important here to remember that  $-\vec{v}$  and  $-1\vec{v}$  initially mean different things! One is a notation for the inverse, the other is the scalar  $-1$  times the vector  $\vec{v}$  it is only after we prove (d) that we may say they are the same!

$$\begin{aligned} -1\vec{v} &= \vec{0} + (-1\vec{v}) \\ &= (-\vec{v} + \vec{v}) + (-1\vec{v}) \\ &= -\vec{v} + (\vec{v} + (-1)\vec{v}) \\ &= -\vec{v} + (1\vec{v} + (-1)\vec{v}) \\ &= -\vec{v} + (1 + (-1))\vec{v} \\ &= -\vec{v} + (0)\vec{v} \\ &= -\vec{v} + \vec{0} \\ &= -\vec{v} \end{aligned}$$