

Section 7.2 Span and Linear Independence

Let $V = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$ be a nonempty set of vectors in a vector space V . The span of this set is denoted $\text{span}\{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$ and is defined to be the set of all linear combinations of the form

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_m \vec{v}_m$$

Where $c_1, c_2, \dots, c_m \in R$

If V consists of infinitely many vectors, then we define $\text{span}(V)$ to be the set of all linear combinations of *finite* subsets of V .

Let $V = \{x^2 + 1, 3x + 2, x^3\}$ determine if $x^3 + 2x^2 + 9x + 8 \in \text{span}(V)$

Let $V = \left\{ \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}, \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ Determine if $\begin{bmatrix} 5 & 0 \\ 6 & -2 \end{bmatrix} \in \text{span}(V)$

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Consider the infinite vector space $V = \{1, x, x^2, x^3 \dots\}$. Given the following equality,

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots$$

Is e^x in the $\text{span}\{V\}$?

Is $x^{6000} - 2x^{500} + 2x + 1$?

Suppose that V is a (possibly infinite) subset of a vector space V , and let $S = \text{span}(V)$. Then S is a subspace of V .

Proof:

Let $\vec{v}_1, \vec{v}_2 \in \text{span}\{V\}$. By the definition of span , $\vec{v}_1 + \vec{v}_2 \in \text{span}\{V\}$ and $r\vec{v}_1 \in V$. Also note that the linear combination $\vec{v}_1 + (-1)\vec{v}_1 = \vec{0} \in \text{span}(V)$.

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Let $\hat{V} = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$ be a set of vectors in a vector space V . Then $\hat{V}$ is linearly independent if the equation

$$c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_m \vec{v}_m = 0$$

has only the trivial solution $c_1 = \dots = c_m = 0$ the set $\hat{V}$ is linearly dependent if there are any nontrivial solutions.

Consider the vector space $V = \{1, x, x^2, x^3, \dots, x^n\}$ is V linearly independent?

Let's think about it, consider

$$c_n x^n + c_{n-1} x^{n-1} + \dots + c_1 x + c_0(1) = 0$$

Suppose that any of the c_i 's are nonzero, will the equation be zero for all possible x values ?

Example: Determine if $\{1, x, e^x\}$ are linearly independent. The trick here is to realize that if there are nonzero coefficients of a linear combination of the vectors that gives us zero, then it must work for all x values. We can therefore choose specific values and see if, for those values, all of the coefficients must be zero to get zero for all chosen x 's. If so, we immediately know there is no choice of non zero coefficients for which we will always get a sum adding to zero that works for every x . Let's choose $x = 0, x = 1, x = 2$

$$c_1(1) + c_2 x + c_3 e^x = 0$$

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octave:2> A=[1 0 1 0; 1 1 exp(1) 0; 1 2 exp(2) 0]
A =
    1.0000    0.0000    1.0000    0.0000
    1.0000    1.0000    2.71828    0.00000
    1.0000    2.0000    7.38906    0.00000

octave:3> rref(A)
ans =
    1    0    0    0
    0    1    0    0
    0    0    1   -0
```

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The set $V = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$ of nonzero vectors is linearly dependent if and only if at least one vector in the set is in the span of the others.

Determine if $\{1, \cos(x), \cos(2x), \sin^2(x)\}$ is a linearly independent subset of $C(R)$.

Let $\hat{V} = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_m\}$ be a subset of a vector space V then:

(a) The set $\hat{V}$ is linearly independent if and only if the equation $c_1\vec{v}_1 + \dots + c_m\vec{v}_m = \vec{v}$ has **at most** one solution for each $\vec{v} \in V$

(b) The set $\hat{V}$ spans V if and only if the equation $c_1\vec{v}_1 + \dots + c_m\vec{v}_m = \vec{v}$ has **at least** one solution for each $\vec{v} \in V$

What might be a set of vectors that span $R^{2 \times 2}$?