

Section 7.3 Basis and Dimension

Let $\hat{V}$ be a subset of a vector space V . The $\hat{V}$ is a basis of V if $\hat{V}$ is linearly independent and spans V

The dimension of a vector space V is the number of vectors in any basis of V , if it has infinitely many vectors, we say the dimension is infinite.

Consider the vector space P , the set of all polynomials. What might be a basis?

In this case, $\hat{V} = \{1, x, x^2, x^3, \dots\}$ is one of many bases for P ! Indeed it is an infinite dimensional one.

$\hat{V}$ spans P because for any degree polynomial in P we need only a finite linear combination of the elements of $\hat{V}$ to get the given polynomial.

$$x^{100} + 2x^{10} + 1 \in P$$

$$\begin{array}{ccccccc} c_{100}x^{100} + c_{99}x^{99} + c_{98}x^{98} + \dots + c_{10}x^{10} + c_1x + c_0 \\ \downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \quad \downarrow \quad \downarrow \quad \downarrow \\ 1 \quad \quad 0 \quad \quad 0 \quad \quad 2 \quad 0 \quad 1 \end{array}$$

Linear independence here is a bit different, we say that the infinite set is linearly independent if every finite subset of that set is.

$$\{1, x, x^2, \dots, x^n\}$$

$$\rightarrow \underline{c_0}(1) + \underline{c_1}(x) + \underline{c_2}x^2 + \dots + \underline{c_n}x^n = 0$$

$$\text{need } c_i = 0$$

Any two polynomials of different degree are linearly independent.

$$\begin{array}{ccccccc} c_1x^3 + c_2x^2 + c_1x & = & \tilde{c}_2x^2 + \tilde{c}_1x + \tilde{c}_0 \\ \downarrow & \downarrow & \downarrow & & \downarrow & \downarrow & \downarrow \\ 0 & 0 & 0 & & 0 & 0 & 0 \end{array}$$

Section 7.3 Basis and Dimension

The set $\vec{V} = \{\vec{v}_1, \dots, \vec{v}_m\}$ is a basis for a vector space V if and only if the equation

$$c_1 \vec{v}_1 + \dots + c_m \vec{v}_m = \vec{v}$$

has a unique solution $c_1, \dots, c_m$ for every $\vec{v} \in V$

Determine if the set $\mathcal{V}$ is a basis for V .

$$\mathcal{V} = \{6x^2 + x - 4, x + 1, -2\}, V = \mathbb{P}^2$$

$$ax^2 + bx + c \in \mathbb{P}^2$$

$$c_1(6x^2 + x - 4) + c_2(x + 1) + c_3(-2) = ax^2 + bx + c$$

$$6c_1x^2 + c_1x - 4c_1 + c_2x + c_2 - 2c_3 = ax^2 + bx + c$$

$$6c_1x^2 + (c_1 + c_2)x - 4c_1 + c_2 - 2c_3 = ax^2 + bx + c$$

$$6c_1 = a \quad c_1 + c_2 = b \quad -4c_1 + c_2 - 2c_3 = c$$

$$6c_1 + 0c_2 + 0c_3 = a$$

$$1c_1 + 1c_2 + 0c_3 = b$$

$$-4c_1 + 1c_2 - 2c_3 = c$$

$$\left[\begin{array}{ccc|c} 6 & 0 & 0 & a \\ 1 & 1 & 0 & b \\ -4 & 1 & -2 & c \end{array} \right]$$

$$c_1 = a/6$$

$$c_1 + c_2 = b$$

$$a/6 + c_2 = b$$

$$c_2 = b - a/6$$

$$-4c_1 + c_2 - 2c_3 = c$$

Section 7.3 Basis and Dimension

We get many of the same theorems about basis and dimension that we had for vectors in $\mathbb{R}^n$!

Suppose that V_1 and V_2 are both basis of a vector space V , then V_1 and V_2 have the same number of elements.

Let $\hat{V} = \{\vec{v}_1, \dots, \vec{v}_m\}$ be a subset of a nontrivial **finite** dimensional vector space V .

(a) If $\hat{V}$ spans V , then either $\hat{V}$ is a basis for V or vectors can be removed from $\hat{V}$ to form a basis for V .

(b) If $\hat{V}$ is linearly independent, then either $\hat{V}$ is a basis for V or vectors can be added to $\hat{V}$ to form a basis for V .

Let $\hat{V} = \{\vec{v}_1, \dots, \vec{v}_m\}$ be a subset of a vector space V with $\dim(V) = n$.

(a) If $m < n$, then $\hat{V}$ does not span V

(b) If $m > n$, then $\hat{V}$ is linearly ~~in~~dependent

$$V = \{1, x, x^3\}$$
$$\hat{V} = \{1, x\}$$

Let V_1 and V_2 be vector spaces with $V_1 \subseteq V_2$ and suppose both have the same definition of addition and scalar multiplication. Then $\dim(V_1) \leq \dim(V_2)$

Let $\hat{V} = \{\vec{v}_1, \dots, \vec{v}_m\}$ be a subset of a vector space V with $\dim(V) = m < \infty$. Then if $\hat{V}$ is linearly independent or spans V , then $\hat{V}$ is a basis for V .

This last theorem says that if we have the same number of vectors in our subset as the dimension of the vector space, we only need to check linear independence or span but not both to determine if it is a basis.

Section 7.3 Basis and Dimension

Assume some knowledge of calculus.

Determine the dimension of the subspace S of $\mathbf{P}$ consisting of polynomials p such that

$$\int_{-1}^1 p(x) dx = 0.$$

What types of polynomials are in this space?

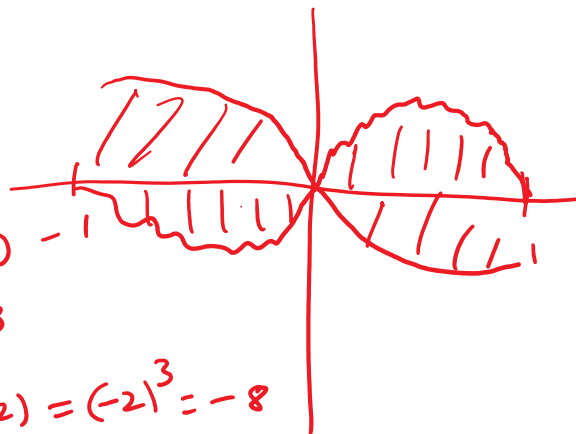
odd polynomials:

$$P(-x) = -P(x)$$

$$P(x) = x^3$$

$$P(-2) = (-2)^3 = -8$$

$$-P(2) = -2^3 = -8$$



What is a basis for those polynomials?

$$\hat{V} = \{ \cancel{1}, \cancel{x}, \cancel{x^2}, x^3, \dots \} ?$$

$$\hat{V} = \{ x, x^3, x^5, \dots \}$$

$$P(x) = x^2 + 1$$

$$\begin{aligned} P(-x) &= (-x)^2 + 1 = x^2 + 1 \\ &= P(x) \end{aligned}$$

What is its dimension?

$$\dim(V) = \infty$$