

Section 8.1 Dot Products and Orthogonal Sets

Back In Section 2.1 we were introduced to the dot product

The dot product: let $\vec{u} \in \mathbb{R}^n$ $\vec{v} \in \mathbb{R}^n$ then

$$\vec{u} \cdot \vec{v} = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ \vdots \\ u_n \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{bmatrix} = u_1 v_1 + u_2 v_2 + u_3 v_3 + \dots + u_n v_n \in \mathbb{R}$$

This product takes two vectors in $\mathbb{R}^n$ to a single number in $\mathbb{R}$. We will focus in on this very important product in chapter 8 when we talk about orthogonality. As it turns out when the dot product of two vectors is zero they are at 90° to each other. It can also be used to determine how much of one vector is in the direction of the other.

$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} = 1(3) + 2(4) = 3 + 8 = 11$$

As we will find out the dot product is an example of mapping from a space $S \rightarrow R$ called an inner product. For now we will focus on the dot product itself, an inner product taking $\mathbb{R}^n \rightarrow \mathbb{R}$.

$$S \rightarrow \mathbb{R}$$

Let's get some properties of the dot product out of the way,

Let $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^n$ and $c \in \mathbb{R}$ then

- (a) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
- (b) $(\vec{u} + \vec{v}) \cdot \vec{w} = (\vec{u} \cdot \vec{w}) + (\vec{v} \cdot \vec{w})$
- (c) $(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (c\vec{v})$
- (d) $\vec{u} \cdot \vec{u} \geq 0$, and $\vec{u} \cdot \vec{u} = 0$ if and only if $\vec{u} = \vec{0}$

Proof of (a)

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \quad \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$$

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

$$\vec{v} \cdot \vec{u} = v_1 u_1 + v_2 u_2 + \dots + v_n u_n$$

Same?

Yes because u_i 's v_i 's $\in \mathbb{R}$

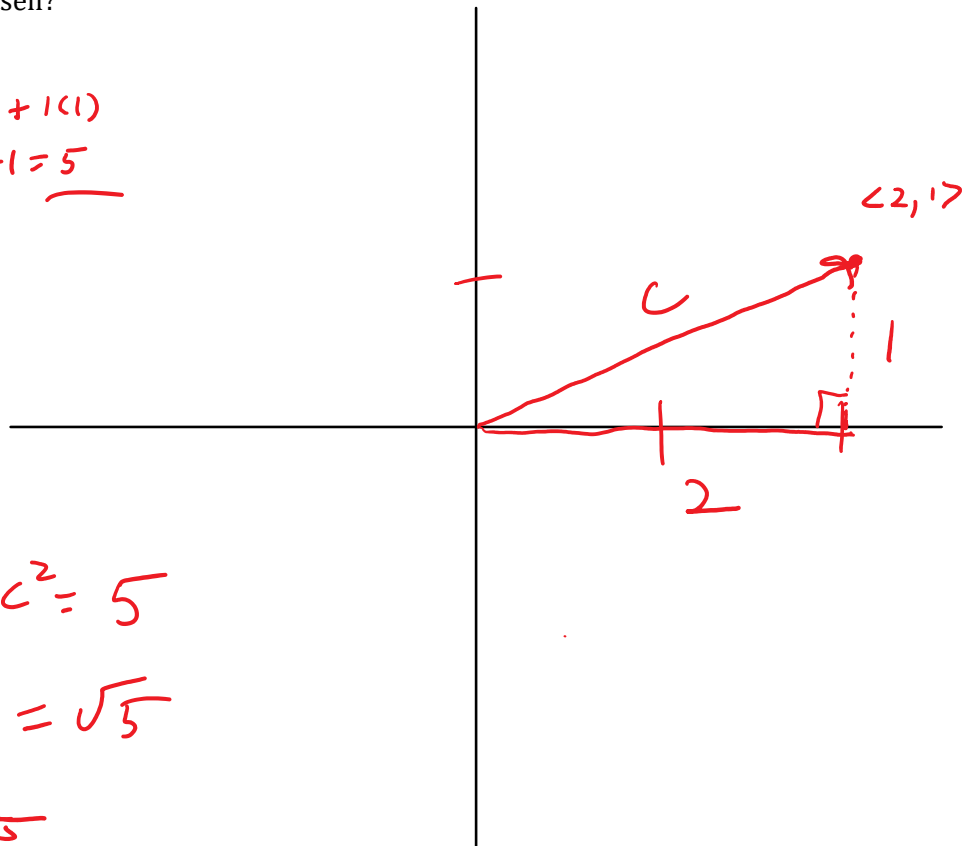
What about $\vec{u} \cdot (\vec{v} \cdot \vec{w}) = (\vec{u} \cdot \vec{v}) \cdot \vec{w}$?

Section 8.1 Dot Products and Orthogonal Sets

Dot product to measure distance

What happens when we dot a vector with itself?

Example: calculate $\vec{u} \cdot \vec{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \end{bmatrix} = 2(2) + 1(1) = 4 + 1 = 5$



$$c^2 = 2^2 + 1^2 = c^2 = 5$$

$$c = \sqrt{5}$$

$$\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}}$$

You might recall from sec 2.1 that the length of a vector is given by:

Magnitude of a vector let $\vec{u} \in \mathbb{R}^n$ then the magnitude is $|\vec{u}| = \sqrt{u_1^2 + u_2^2 + u_3^2 \cdots u_n^2}$

In fact:

$$\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \Rightarrow \vec{u} \cdot \vec{u} = u_1 u_1 + u_2 u_2 + \cdots + u_n u_n = u_1^2 + u_2^2 + \cdots + u_n^2$$

$$|\vec{u}| = \sqrt{\vec{u} \cdot \vec{u}}$$

We call this the norm of $\vec{u}$

Let $\vec{u} \in \mathbb{R}^n$ then the norm(or length of) $\vec{u}$ is given by

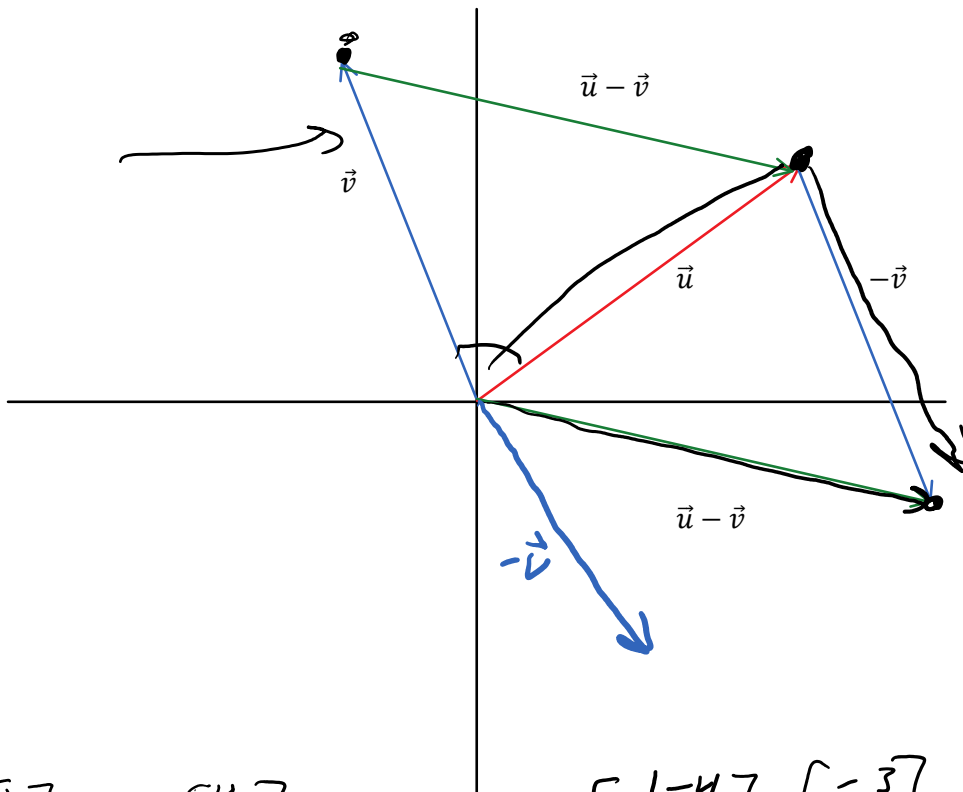
$$\|\vec{u}\| = \sqrt{\vec{u} \cdot \vec{u}}$$

Section 8.1 Dot Products and Orthogonal Sets

Distance between two vectors

Let $\vec{u}, \vec{v} \in R^n$ then the distance between $\vec{u}$ and $\vec{v}$ is given by

$$d(\vec{u}, \vec{v}) = \|\vec{u} - \vec{v}\|$$



$$\vec{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad \vec{v} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \quad \vec{u} - \vec{v} = \begin{bmatrix} 1-4 \\ 2-5 \\ 3-6 \end{bmatrix} = \begin{bmatrix} -3 \\ -3 \\ -3 \end{bmatrix} \quad \frac{\|\vec{u} - \vec{v}\|}{\sqrt{(\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v})}}$$

The dot product can also tell us the angle between two vectors.

$$= \sqrt{(-3)^2 + (-3)^2 + (-3)^2} = \sqrt{27}$$

Let $\vec{u}, \vec{v} \in R^n$ then

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos(\theta)$$

$$\cos(\theta) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$$

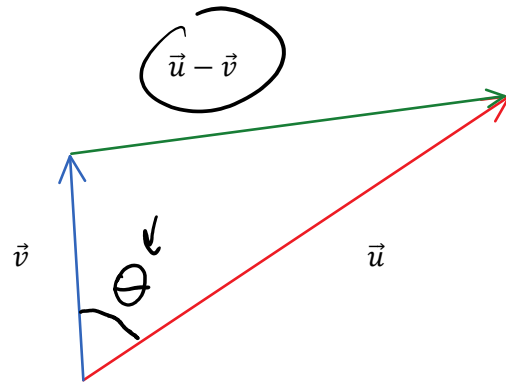
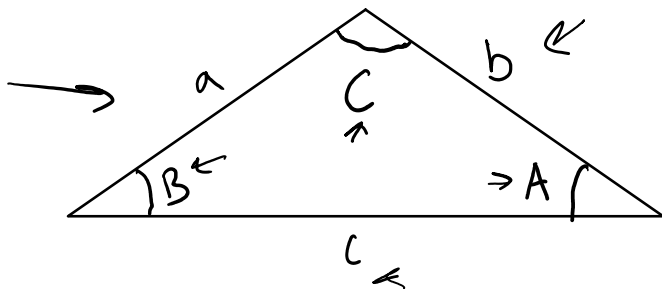
Section 8.1 Dot Products and Orthogonal Sets

Let $\vec{u}, \vec{v} \in R^n$ then

$$\vec{u} \cdot \vec{v} = ||u|| ||v|| \cos(\theta)$$

Proof: Let $\vec{u}, \vec{v} \in R^n$

Think about the fact that they form a triangle in a plane.



Law of Cosines: $c^2 = a^2 + b^2 - 2ab \cos(C)$

Law of Cosines: $||\vec{u} - \vec{v}||^2 = ||\vec{u}||^2 + ||\vec{v}||^2 - 2 ||\vec{u}|| ||\vec{v}|| \cos(\theta)$

$$||\vec{u} - \vec{v}||^2 = ||\vec{u}||^2 + ||\vec{v}||^2 + 2 ||\vec{u}|| ||\vec{v}|| \cos(\theta)$$

$$(\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) = \vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v} + 2(\vec{u} \cdot \vec{v}) \cos(\theta)$$

$$\vec{u} \cdot \vec{u} - (\vec{u} \cdot \vec{v}) - (\vec{v} \cdot \vec{u}) + \vec{v} \cdot \vec{v} = \vec{u} \cdot \vec{u} + \vec{v} \cdot \vec{v} - 2(\vec{u} \cdot \vec{v}) \cos(\theta)$$

$$-(\vec{u} \cdot \vec{v}) - (\vec{v} \cdot \vec{u}) = -2(\vec{u} \cdot \vec{v}) \cos(\theta)$$

$$-2(\vec{u} \cdot \vec{v}) = -2(\vec{u} \cdot \vec{v}) \cos(\theta)$$

$$\vec{u} \cdot \vec{v} = ||u|| ||v|| \cos(\theta)$$

So what happens when the angle between them is 90° ?

$$\vec{u} \cdot \vec{v} = ||\vec{u}|| ||\vec{v}|| \cos(90) = ||\vec{u}|| ||\vec{v}|| (0) = 0$$

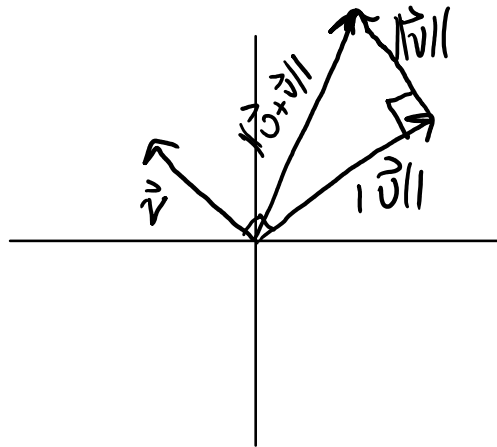
Section 8.1 Dot Products and Orthogonal Sets

Vectors $\vec{u}, \vec{v} \in \mathbb{R}^n$ are orthogonal if and only if

$$\vec{u} \cdot \vec{v} = 0$$

Pythagorean theorem

Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ then $\|\vec{u} + \vec{v}\|^2 = \|\vec{u}\|^2 + \|\vec{v}\|^2$ if and only if $\vec{u} \cdot \vec{v} = 0$



Cauchy – Schwarz inequality : if $\vec{u}, \vec{v} \in \mathbb{R}^n$ then

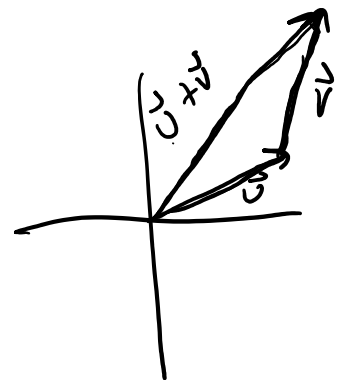
$$|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \|\vec{v}\|$$

Proving one of these would make a great quiz problem!

The proofs are easily found online.

Triangle Inequality: if $\vec{u}, \vec{v} \in \mathbb{R}^n$ then

$$\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$$



Section 8.1 Dot Products and Orthogonal Sets

Let S be a subspace of $\mathbb{R}^n$. We say that a vector $\vec{u} \in \mathbb{R}^n$ is orthogonal to S if $\vec{u} \cdot \vec{s} = 0$ for every $\vec{s} \in S$. The set of all such vectors is called the orthogonal complement of S and is denoted $S^\perp$. $S^\perp$ is perp

An easy example to visualize is that of a vector which is perpendicular to a plane. In fact that vector can define the plane.

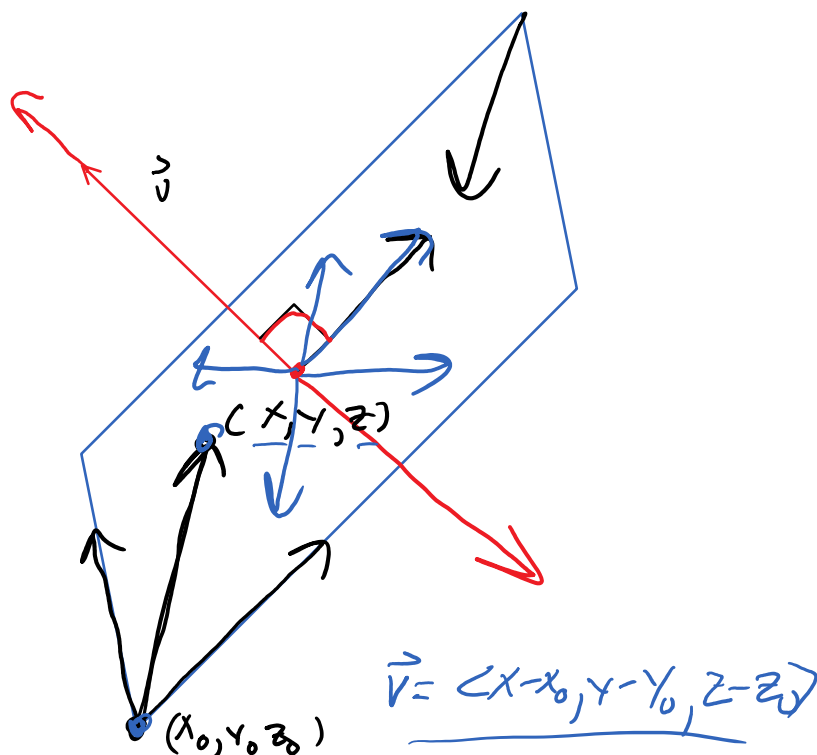
Let $\vec{u} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$ then the equation of the plane perpendicular to $\vec{u}$ going containing a point (x_0, y_0, z_0) is given by:

$$\rightarrow a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$$

Which is the same as the requirement

$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \cdot \begin{bmatrix} x - x_0 \\ y - y_0 \\ z - z_0 \end{bmatrix} = 0$ that is, that $\vec{u}$ is perpendicular to every vector in the plane.

$$S = \text{span} \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right\}$$



If S is a subspace of $\mathbb{R}^n$ then so is $S^\perp$.

Proof:

$$\begin{aligned} (1) \quad & \vec{u}, \vec{v} \in S^\perp \text{ then so is } \vec{u} + \vec{v} \\ & \downarrow \quad \downarrow \\ & \forall \vec{s} \in S \quad \vec{u} \cdot \vec{s} = 0 \quad \vec{v} \cdot \vec{s} = 0 \\ & \quad \quad \quad (\vec{u} + \vec{v}) \cdot \vec{s} = ? \\ & \quad \quad \quad = \vec{u} \cdot \vec{s} + \vec{v} \cdot \vec{s} = 0 + 0 = 0 \\ (2) \quad & \vec{u} \in S^\perp \text{ then so is } c\vec{u} \\ & \quad \quad \quad (c\vec{u}) \cdot \vec{s} = ? = c(\vec{u} \cdot \vec{s}) = c(0) = 0 \end{aligned}$$

③ $\vec{0} \in S^\perp$ yes because $\vec{0} \cdot \vec{s} = 0 \quad \forall \vec{s} \in S$

Section 8.1 Dot Products and Orthogonal Sets

Let $V = \{\vec{s}_1, \dots, \vec{s}_k\}$ be a basis for a subspace S and let $\vec{u}$ be a vector. Then $\vec{u} \in S^\perp$ if and only if

$$\vec{u} \cdot \vec{s}_1 = \vec{u} \cdot \vec{s}_2 = \dots = \vec{u} \cdot \vec{s}_k = 0$$

Proof: Suppose $\vec{u} \in S^\perp$, then by definition it is orthogonal to all vectors in S and thus orthogonal to the basis vectors.

Now suppose $\vec{u} \cdot \vec{s}_1 = \vec{u} \cdot \vec{s}_2 = \dots = \vec{u} \cdot \vec{s}_k = 0$ and let $\vec{s} \in S$ then because V is a basis we can say,

$$\vec{s} = c_1 \vec{s}_1 + c_2 \vec{s}_2 + \dots + c_k \vec{s}_k$$

And then

$$\begin{aligned} \vec{u} \cdot \vec{s} &= \vec{u} \cdot (c_1 \vec{s}_1 + c_2 \vec{s}_2 + \dots + c_k \vec{s}_k) = \vec{u} \cdot c_1 \vec{s}_1 + \vec{u} \cdot c_2 \vec{s}_2 + \dots + \vec{u} \cdot c_k \vec{s}_k = c_1 \vec{u} \cdot \vec{s}_1 + c_2 \vec{u} \cdot \vec{s}_2 + \dots + c_k \vec{u} \cdot \vec{s}_k \\ &= c_1(0) + c_2(0) + \dots + c_k(0) = 0 \end{aligned}$$

So for every $\vec{s} \in S$ $\vec{u} \cdot \vec{s} = 0$ thus $\vec{u} \in S^\perp$.

A set of vectors V in R^n form an orthogonal set if and only if $\vec{v}_i \cdot \vec{v}_j = 0$ whenever $i \neq j$ for all vectors in V . Another way to state this is $\vec{v}_i \cdot \vec{v}_j = \delta_{ij} \|\vec{v}_i\|^2$ $\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

An orthogonal set of vectors $V = \{\vec{v}_1, \dots, \vec{v}_k\}$ is linearly independent

Proof:

Let $\{\vec{v}_1, \dots, \vec{v}_k\}$ be an orthogonal set of vectors.

$$\vec{c}_1 \vec{v}_1 + \dots + \vec{c}_k \vec{v}_k = \vec{0} \Rightarrow c_1 = c_2 = \dots = c_k = 0 \quad \text{LI}$$

$$\vec{v}_1 \cdot (c_1 \vec{v}_1 + \dots + c_k \vec{v}_k) = \vec{v}_1 \cdot \vec{0}$$

$$c_1 \vec{v}_1 \cdot \vec{v}_1 + c_2 \vec{v}_1 \cdot \vec{v}_2 + \dots + c_k \vec{v}_1 \cdot \vec{v}_k = 0$$

Consider

$$c_1 \|\vec{v}_1\|^2 + c_2(0) + \dots + c_k(0) = 0$$

$$c_1 \|\vec{v}_1\|^2 = 0 \Rightarrow c_1 = 0$$

So since $\vec{v}_1 \neq \vec{0}$, $\|\vec{v}_1\|^2 \neq 0$ thus $c_1 = 0$. Repeat with all of the vectors in V to find that indeed if

$c_1 \vec{v}_1 + \dots + c_k \vec{v}_k = \vec{0}$ then it must be that $c_1 = c_2 = \dots = c_k = 0$ the the vectors are linearly independent.

$$c_1(0) + c_2 \|\vec{v}_2\|^2 + c_3(0) + \dots + c_k(0) = 0$$

$$c_2 \|\vec{v}_2\|^2 = 0$$

Section 8.1 Dot Products and Orthogonal Sets

Let $V = \{\vec{v}_1 \dots \vec{v}_k\}$ be an orthogonal basis for a subspace S , then for any vector $\vec{s} \in S$ we can write $\vec{s}$ as

$$\vec{s} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_k \vec{v}_k \text{ where } c_i = \frac{\vec{v}_i \cdot \vec{s}}{\|\vec{v}_i\|^2}$$

Proof: Again consider $\vec{v}_1 \cdot \vec{s} = \vec{v}_1 \cdot (c_1 \vec{v}_1 + \dots + c_k \vec{v}_k)$ and let's see if we can solve for c_1 .

$$\vec{v}_1 \cdot \vec{s} = \vec{v}_1 \cdot (c_1 \vec{v}_1) + \vec{v}_1 \cdot (c_2 \vec{v}_2) + \dots + \vec{v}_1 \cdot (c_k \vec{v}_k) = \vec{v}_1 \cdot \vec{s}$$

$$\Rightarrow \vec{v}_1 \cdot \vec{s} = c_1 \underbrace{\vec{v}_1 \cdot \vec{v}_1}_{\|\vec{v}_1\|^2} + c_2 \underbrace{(\vec{v}_1 \cdot \vec{v}_2)}_0 + \dots + c_k \underbrace{(\vec{v}_1 \cdot \vec{v}_k)}_0$$

$$\Rightarrow \vec{v}_1 \cdot \vec{s} = c_1 \|\vec{v}_1\|^2$$

$$c_1 = \frac{\vec{v}_1 \cdot \vec{s}}{\|\vec{v}_1\|^2}$$

for c_2

$$\vec{v}_2 \cdot \vec{s} = c_1 \cancel{\vec{v}_2 \cdot \vec{v}_1} + c_2 \vec{v}_2 \cdot \vec{v}_2 + \dots + c_k \cancel{\vec{v}_2 \cdot \vec{v}_k}$$

$$\vec{v}_2 \cdot \vec{s} = c_2 \|\vec{v}_2\|^2$$

$$c_2 = \frac{\vec{v}_2 \cdot \vec{s}}{\|\vec{v}_2\|^2}$$

By

Section 8.1 Dot Products and Orthogonal Sets

Find a basis for $S^\perp$ for the subspace S .

$$S = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ -7 \end{bmatrix} \right\}$$

8a

$$\left\{ \begin{bmatrix} \boxed{} \\ \boxed{} \\ \boxed{} \end{bmatrix} \right\}$$

$$\rightarrow [-1, 7, 1, 0, 0, 1]$$

$$\dim(S^\perp) = 2$$

$$\text{span} \left\{ \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}, \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} \right\} = S^\perp$$

$$\begin{bmatrix} 1 \\ 1 \\ -7 \end{bmatrix} \cdot \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} = 0$$

$$\begin{bmatrix} 1 \\ 1 \\ -7 \end{bmatrix} \cdot \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} = 0$$

$$a_1 + b_1 - 7c_1 = 0$$

$$a_1 = 7, b_1 = 0, c_1 = 1$$

$$7 + 0 - 7 = 0 \checkmark$$

$$c_1 \begin{bmatrix} 7 \\ 0 \\ 1 \end{bmatrix}$$

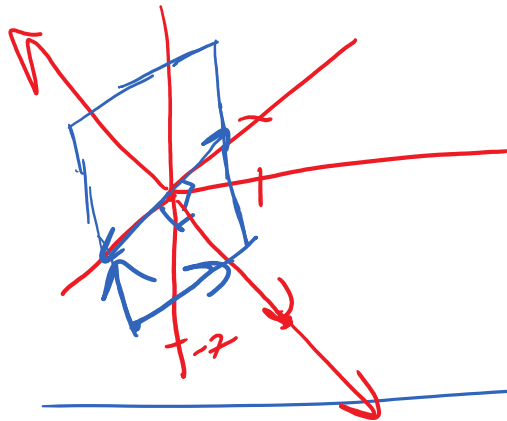
$$a_2 + b_2 - 7c_2 = 0$$

$$\text{choose } a_2 = 0$$

$$0 + b_2 - 7c_2 = 0$$

$$b_2 = 7, c_2 = 1$$

$$\rightarrow \begin{bmatrix} 0 \\ 7 \\ 1 \end{bmatrix}$$



let V be a vector space
 $\dim(V) = n$ and S be
 a subspace of V
 Then $\dim(S) + \dim(S^\perp) = \dim(V)$
 $= n$

Section 8.1 Dot Products and Orthogonal Sets

Find a basis for $S^\perp$ for the subspace S .

$$S = \text{span} \left\{ \begin{bmatrix} 1 \\ 1 \\ -7 \end{bmatrix} \right\}$$

A More Methodical way now that my brain is working!

8a

$$\left\{ \begin{bmatrix} \boxed{} \\ \boxed{} \\ \boxed{} \end{bmatrix} \right\}$$

$$\text{key } [-1, 7; 1, 0; 0, 1]$$

we know that $\dim(S^\perp)$ needs to be 2
 so that $\dim(S) + \dim(S^\perp) = 3$
 so we need 2 vectors to span $S^\perp$
 The vectors we want need to satisfy

$$\begin{bmatrix} 1 \\ 1 \\ -7 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0 \quad \text{we can write this as a matrix equation}$$

$1 \times 3 \quad 3 \times 1 \quad 1 \times 1$

$$\begin{bmatrix} 1 & 1 & -7 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = 0$$

So we want the null space of $\begin{bmatrix} 1 & 1 & -7 \end{bmatrix}$!

$$\begin{bmatrix} 1 & 1 & -7 & | & 0 \end{bmatrix} \quad \text{note only a pivot in a position so } b, c \text{ are free variables!}$$

$$\text{let } c = t \quad b = s \quad \text{then}$$

$$\begin{bmatrix} 1 & 1 & -7 & | & 0 \end{bmatrix} \Rightarrow a + b - 7c = 0 \Rightarrow a + t - 7s = 0$$

$$\Rightarrow a = -t + 7s$$

$$\text{so } \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} -t + 7s \\ s \\ t \end{bmatrix} = \begin{bmatrix} -t \\ 0 \\ t \end{bmatrix} + \begin{bmatrix} 7s \\ s \\ 0 \end{bmatrix} = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + s \begin{bmatrix} 7 \\ 1 \\ 0 \end{bmatrix}$$

$$\text{so our null space} = S^\perp = \text{span} \left\{ \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 7 \\ 1 \\ 0 \end{bmatrix} \right\}$$