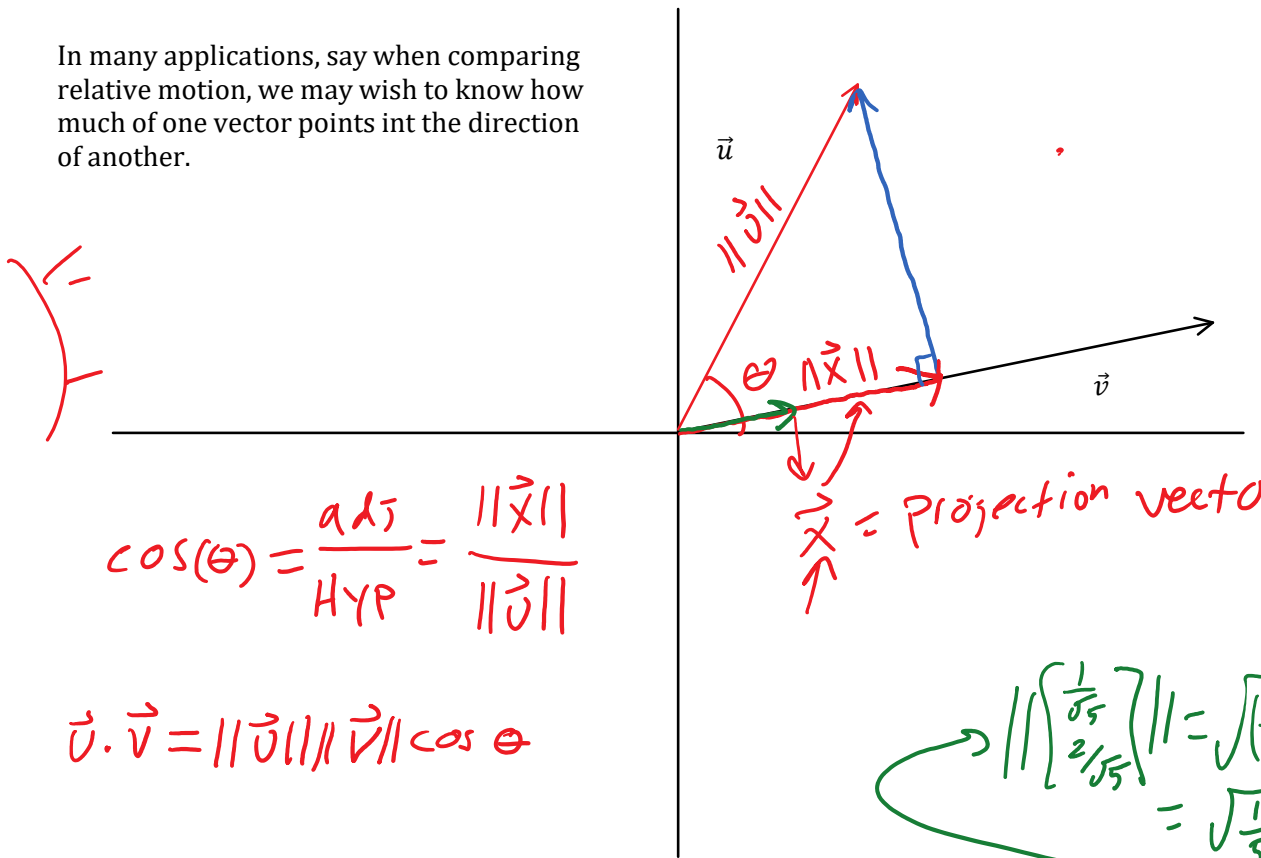


Section 8.2 Projection and the Gram-Schmidt Process

In the last section we talked about an inner product on $\mathbb{R}^n$, namely the dot product. In this section we will talk about some uses for this product. In particular, measuring how much of one vector is in the direction of another and a method for taking a given basis for a space and finding a new basis made of orthogonal vectors.

In many applications, say when comparing relative motion, we may wish to know how much of one vector points in the direction of another.



$$\cos(\theta) = \frac{\text{adj}}{\text{hyp}} = \frac{\|\vec{x}\|}{\|\vec{u}\|}$$

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \theta$$

$$\left\| \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix} \right\| = \sqrt{\left(\frac{1}{\sqrt{5}}\right)^2 + \left(\frac{2}{\sqrt{5}}\right)^2} = \sqrt{\frac{1}{5} + \frac{4}{5}} = \sqrt{\frac{5}{5}} = \sqrt{1} = 1$$

Let's find the projection vector above. We will obtain it in the following way.

1. Find a vector of length 1 in the direction of $\vec{v}$

$$\frac{\vec{v}}{\|\vec{v}\|}$$

$$\left\| \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

$$\frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}$$

2. Find the length of the projection vector.

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \theta$$

$$\cos(\theta) = \frac{\|\vec{x}\|}{\|\vec{u}\|}$$

$$\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$$

$$\frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} = \frac{\|\vec{x}\|}{\|\vec{u}\|} \Rightarrow \|\vec{x}\| = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} \|\vec{u}\|$$

$$\boxed{\|\vec{x}\| = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|}}$$

3. Scale the unit vector by the length from 2.

$$\vec{x} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|} \frac{\vec{v}}{\|\vec{v}\|} = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v}$$

Section 8.2 Projection and the Gram-Schmidt Process

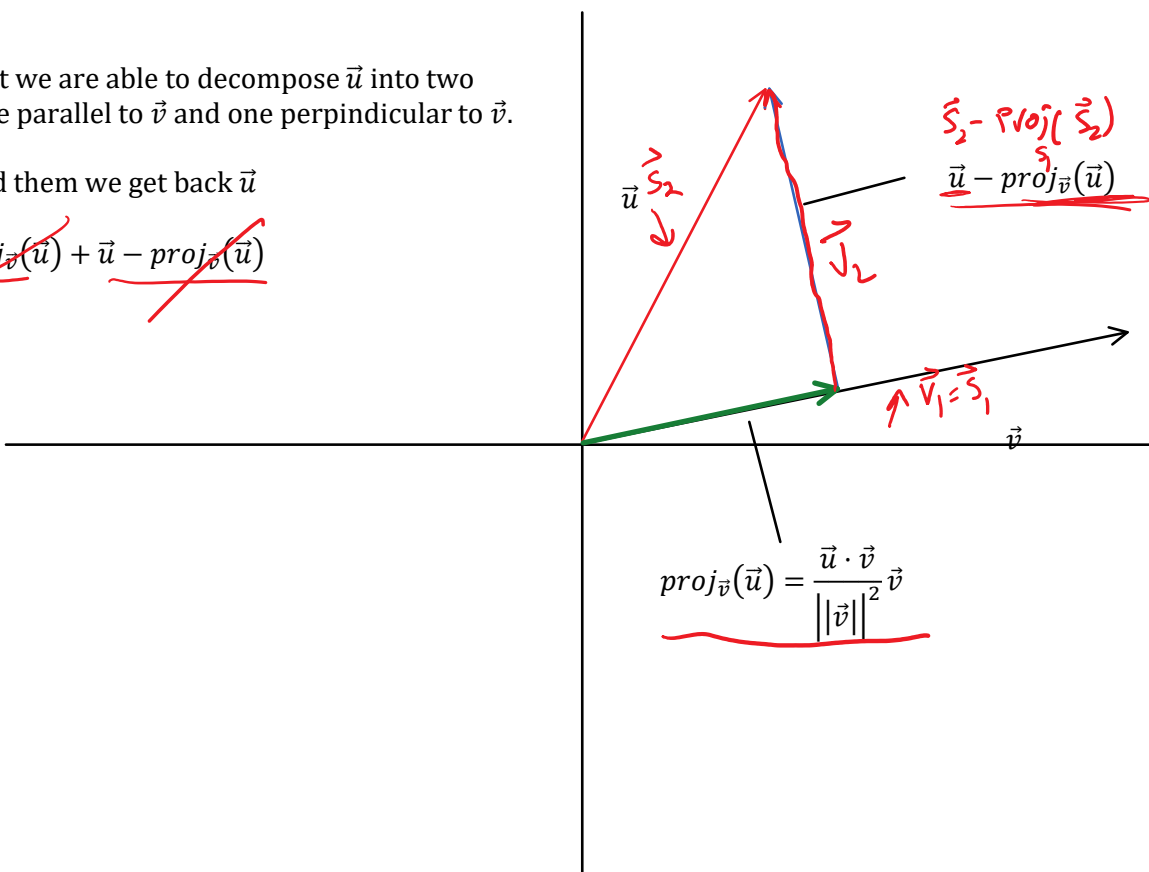
Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ with $\vec{v} \neq \vec{0}$. The projection of $\vec{u}$ onto $\vec{v}$ is defined as

$$\vec{x} = \text{proj}_{\vec{v}}(\vec{u}) = \frac{\vec{u} \cdot \vec{v}}{\|\vec{v}\|^2} \vec{v}$$

Note that we are able to decompose $\vec{u}$ into two parts one parallel to $\vec{v}$ and one perpendicular to $\vec{v}$.

If we add them we get back $\vec{u}$

$$\vec{u} = \text{proj}_{\vec{v}}(\vec{u}) + \vec{u} - \text{proj}_{\vec{v}}(\vec{u})$$



So we can measure how much of one vector points in the direction of another, and further we can remove that from the original vector to obtain one which is perpendicular to the vector we were projection onto. This means given two vectors we can always find from those, two other vectors which are perpendicular to each other.

This might make one wonder, if I have a set of linearly independent vectors $S = \{s_1 \dots s_k\}$ which form a basis of a subspace, which are not orthogonal, can I use them to find a basis for the same space which is orthogonal? After all, we know that orthogonal vectors are already linearly independent so a list of them is a basis for some subspace.

Section 8.2 Projection and the Gram-Schmidt Process

Let's suppose that I have some linearly independent vectors for a subspace $S = \text{span}\{\vec{s}_1 \dots \vec{s}_k\}$

Let's start with $\vec{s}_1$ and rename it $\vec{v}_1$

$$\vec{v}_1 = \vec{s}_1$$

Next find a vector that is orthogonal to it using $\vec{s}_2$ by removing the bit of $\vec{v}_1$ that is in the direction of $\vec{s}_2$. Call the result $\vec{v}_2$.

$$\vec{v}_2 = \vec{s}_2 - \text{proj}_{\vec{v}_1}(\vec{s}_2)$$

$\vec{v}_2 \cdot \vec{v}_1 = 0$

So $\vec{v}_2$ is orthogonal to $\vec{v}_1$, great! Now let's find a vector that is orthogonal to both $\vec{v}_1$ and $\vec{v}_2$ using $\vec{s}_3$ by removing the parts of $\vec{s}_3$ that are in the direction of each. Call it $\vec{v}_3$.

$$\vec{v}_3 = \vec{s}_3 - \text{proj}_{\vec{v}_1}(\vec{s}_3) - \text{proj}_{\vec{v}_2}(\vec{s}_3)$$

$\vec{v}_3 \cdot \vec{v}_1 = 0$
 $\vec{v}_3 \cdot \vec{v}_2 = 0$

So, $\vec{v}_3$ is orthogonal to both $\vec{v}_1$ and $\vec{v}_2$! We can continue finding orthogonal vectors in the same way.

$$\begin{aligned} \vec{v}_4 &= \vec{s}_4 - \text{proj}_{\vec{v}_1}(\vec{s}_4) - \text{proj}_{\vec{v}_2}(\vec{s}_4) - \text{proj}_{\vec{v}_3}(\vec{s}_4) \\ &\vdots \\ \vec{v}_k &= \vec{s}_k - \text{proj}_{\vec{v}_1}(\vec{s}_k) - \text{proj}_{\vec{v}_2}(\vec{s}_k) - \text{proj}_{\vec{v}_3}(\vec{s}_k) - \dots - \text{proj}_{\vec{v}_{k-1}}(\vec{s}_k) \end{aligned}$$

By design, because we keep removing parts that are in the direction of others the set $V = \{\vec{v}_1 \dots \vec{v}_k\}$ will be orthogonal and therefore linearly independent. Because of the independence it is a basis for some subspace, the question is, is that subspace the same as our original space S ?

We can prove that they are the same by showing

$$\text{span}\{\vec{s}_1 \dots \vec{s}_k\} \subseteq \text{span}\{\vec{v}_1 \dots \vec{v}_k\}$$

That is if the span of the $\vec{s}_i$'s is contained in that of the $\vec{v}_i$'s then they have to be the same because they have the same number of vectors. To see this let me rewrite $\vec{v}_i$ and solve it for $\vec{s}_i$

$$\begin{aligned} \vec{v}_i &= \vec{s}_i - \frac{\vec{s}_i \cdot \vec{v}_1}{\|\vec{v}_1\|^2} \vec{v}_1 - \frac{\vec{s}_i \cdot \vec{v}_2}{\|\vec{v}_2\|^2} \vec{v}_2 - \dots - \frac{\vec{s}_i \cdot \vec{v}_{i-1}}{\|\vec{v}_{i-1}\|^2} \vec{v}_{i-1} \\ \vec{s}_i &= \vec{v}_i + \underbrace{\frac{\vec{s}_i \cdot \vec{v}_1}{\|\vec{v}_1\|^2}}_{c_1} \vec{v}_1 + \underbrace{\frac{\vec{s}_i \cdot \vec{v}_2}{\|\vec{v}_2\|^2}}_{c_2} \vec{v}_2 + \dots + \underbrace{\frac{\vec{s}_i \cdot \vec{v}_{i-1}}{\|\vec{v}_{i-1}\|^2}}_{c_{i-1}} \vec{v}_{i-1} \end{aligned}$$

$\text{proj}_{\vec{v}_j}(\vec{s}_i)$

Section 8.2 Projection and the Gram-Schmidt Process

$$\text{So } \vec{s}_i = \vec{v}_i + \frac{\vec{s}_i \cdot \vec{v}_1}{\|\vec{v}_1\|^2} \vec{v}_1 + \frac{\vec{s}_i \cdot \vec{v}_2}{\|\vec{v}_2\|^2} \vec{v}_2 + \dots + \frac{\vec{s}_i \cdot \vec{v}_{i-1}}{\|\vec{v}_{i-1}\|^2} \vec{v}_{i-1}$$

Let $\vec{s} \in S$ be given as $\vec{s} = c_1 \vec{s}_1 + c_2 \vec{s}_2 + \dots + c_k \vec{s}_k$ can we find $c'_1, c'_2, \dots, c'_k$ such that

$$\vec{s} = c'_1 \vec{v}_1 + c'_2 \vec{v}_2 + \dots + c'_k \vec{v}_k??$$

Starting with $\vec{s} = c_1 \vec{s}_1 + c_2 \vec{s}_2 + \dots + c_k \vec{s}_k$ Let's just write the s vectors in terms of the v vectors!

$$\vec{s} = c_1 (\vec{v}_1) + c_2 \left(\vec{v}_2 + \frac{\vec{s}_2 \cdot \vec{v}_1}{\|\vec{v}_1\|^2} \vec{v}_1 \right) + \dots + c_k \left(\vec{v}_k + \frac{\vec{s}_k \cdot \vec{v}_1}{\|\vec{v}_1\|^2} \vec{v}_1 + \frac{\vec{s}_k \cdot \vec{v}_2}{\|\vec{v}_2\|^2} \vec{v}_2 + \dots + \frac{\vec{s}_k \cdot \vec{v}_{k-1}}{\|\vec{v}_{k-1}\|^2} \vec{v}_{k-1} \right)$$

Right here we know there is a linear combination of the $\vec{v}$'s that get us to $\vec{s}$! So we can say that since we can reach every vector $\vec{s} \in S$ using the orthogonal vectors $\{v_1 \dots v_k\}$ that,

$$\text{span}\{s_1 \dots s_k\} \subseteq \text{span}\{v_1 \dots v_k\}$$

And because both sets have the same number of vectors, there cannot be any other vectors we can reach with $\{v_1 \dots v_k\}$ that we can't with $\{s_1 \dots s_k\}$ so it must be that

$$\text{span}\{s_1 \dots s_k\} = \text{span}\{v_1 \dots v_k\}$$

And so we now have a set of orthogonal vectors $\{v_1 \dots v_k\}$ which span the exact same set as $\{s_1 \dots s_k\}$

Gram-Schmidt process. Let $\{s_1 \dots s_k\}$ be a basis for a subspace S . An orthogonal basis, $\{v_1 \dots v_k\}$, can be found for S and is calculated as follows.

$$\begin{aligned} \vec{v}_1 &= \vec{s}_1 \\ \vec{v}_2 &= \vec{s}_2 - \text{proj}_{\vec{v}_1}(\vec{s}_2) \\ \vec{v}_3 &= \vec{s}_3 - \text{proj}_{\vec{v}_1}(\vec{s}_3) - \text{proj}_{\vec{v}_2}(\vec{s}_3) \\ \vec{v}_4 &= \vec{s}_4 - \text{proj}_{\vec{v}_1}(\vec{s}_4) - \text{proj}_{\vec{v}_2}(\vec{s}_4) - \text{proj}_{\vec{v}_3}(\vec{s}_4) \\ &\vdots \\ \vec{v}_k &= \vec{s}_k - \text{proj}_{\vec{v}_1}(\vec{s}_k) - \text{proj}_{\vec{v}_2}(\vec{s}_k) - \text{proj}_{\vec{v}_3}(\vec{s}_k) - \text{proj}_{\vec{v}_4}(\vec{s}_k) - \dots - \text{proj}_{\vec{v}_{k-1}}(\vec{s}_k) \end{aligned}$$

Section 8.2 Projection and the Gram-Schmidt Process

Let's visualize this:

<https://www.youtube.com/watch?v=Ys28-Yq21B8>

Apply the Gram-Schmidt process to find an orthogonal basis for S .

$$S = \text{span} \left\{ \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 8 \\ 4 \end{bmatrix} \right\}$$

3a

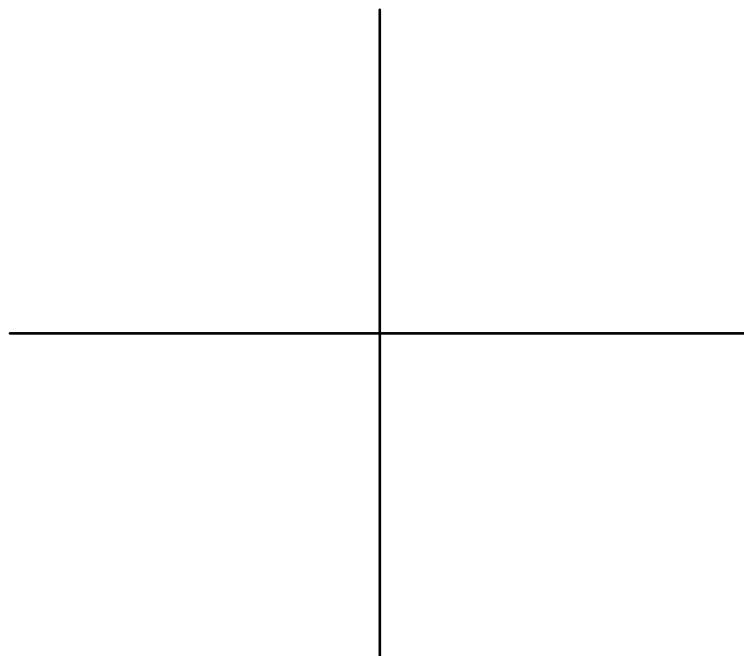
$$\left\{ \begin{bmatrix} \boxed{} \\ \boxed{} \end{bmatrix} \right\}$$



 $[3, 2; -1, 6]$

Gram-Schmidt process. Let $\{s_1 \dots s_k\}$ be a basis for a subspace S . An orthogonal basis, $\{v_1 \dots v_k\}$, can be found for S and is calculated as follows.

$$\begin{aligned} \vec{v}_1 &= \vec{s}_1 \\ \vec{v}_2 &= \vec{s}_2 - \text{proj}_{\vec{v}_1}(\vec{s}_2) \\ \vec{v}_3 &= \vec{s}_3 - \text{proj}_{\vec{v}_1}(\vec{s}_3) - \text{proj}_{\vec{v}_2}(\vec{s}_3) \\ \vec{v}_4 &= \vec{s}_4 - \text{proj}_{\vec{v}_1}(\vec{s}_4) - \text{proj}_{\vec{v}_2}(\vec{s}_4) - \text{proj}_{\vec{v}_3}(\vec{s}_4) \\ &\vdots \\ \vec{v}_k &= \vec{s}_k - \text{proj}_{\vec{v}_1}(\vec{s}_k) - \text{proj}_{\vec{v}_2}(\vec{s}_k) - \text{proj}_{\vec{v}_3}(\vec{s}_k) - \text{proj}_{\vec{v}_4}(\vec{s}_k) - \dots - \text{proj}_{\vec{v}_{k-1}}(\vec{s}_k) \end{aligned}$$



Section 8.2 Projection and the Gram-Schmidt Process

Another modification we can make to our basis is that of normalization, which just means we make all of the vectors have length 1. An orthogonal basis for which the length of each vector is 1 is called "**orthonormal**".

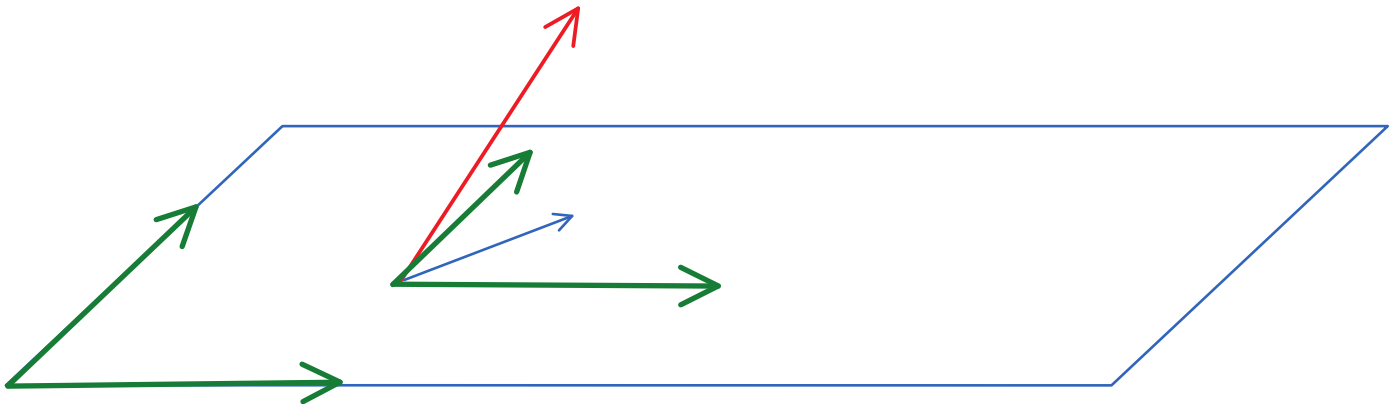
A set of vectors $\{\vec{w}_1 \dots \vec{w}_k\}$ is orthonormal if they are orthogonal and $\|\vec{w}_i\| = 1$ for each vector in the set.

If we want to normalize our basis from the previous example we can just divide each by its own magnitude.

$$\left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 6 \end{bmatrix} \right\}$$

Projection onto a Subspace: Let S be a nonzero subspace with orthogonal basis $\{\vec{v}_1 \dots \vec{v}_k\}$. Then the projection of a vector $\vec{u}$ onto S is given by,

$$\text{proj}_S \vec{u} = \frac{\vec{v}_1 \cdot \vec{u}}{\|\vec{v}_1\|^2} \vec{v}_1 + \frac{\vec{v}_2 \cdot \vec{u}}{\|\vec{v}_2\|^2} \vec{v}_2 + \dots + \frac{\vec{v}_k \cdot \vec{u}}{\|\vec{v}_k\|^2} \vec{v}_k$$



Let S be a non zero subspace with orthogonal basis $\{\vec{v}_1 \dots \vec{v}_k\}$, and let $\vec{u} \in \mathbb{R}^n$. Then,

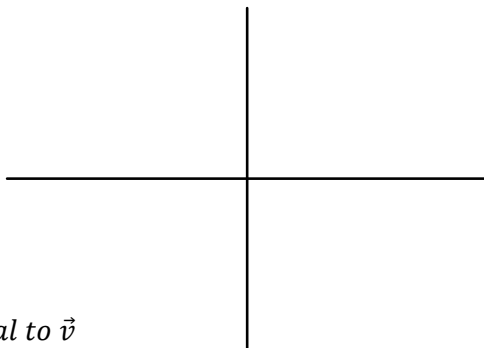
- (a) $\text{proj}_S \vec{u}$ is in S .
- (b) $\vec{u} - \text{proj}_S \vec{u}$ is orthogonal to S .
- (c) if $\vec{u}$ is in S then $\text{proj}_S \vec{u} = \vec{u}$.
- (d) $\text{proj}_S \vec{u}$ is independent of the choice of orthogonal basis for S .

Section 8.2 Projection and the Gram-Schmidt Process

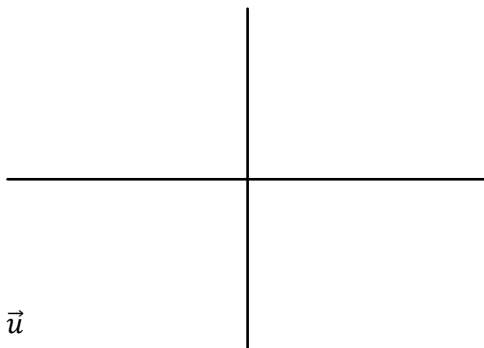
Summary of some projection properties

Let $\vec{u}, \vec{v} \in \mathbb{R}^n$ with v nonzero and $c \in \mathbb{R}$ also nonzero. Then,

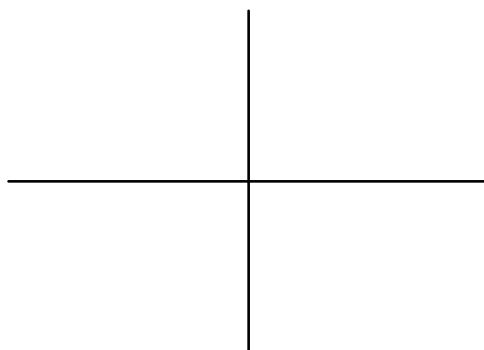
(a) $\text{proj}_{\vec{v}} \vec{u} \in \text{span}\{\vec{v}\}$



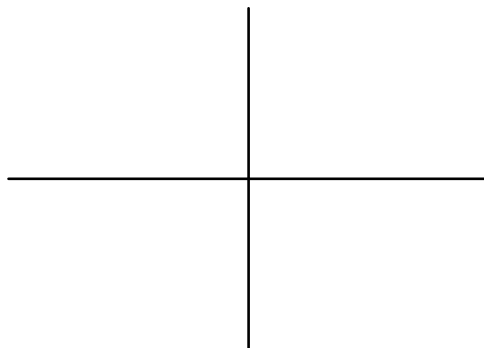
(b) $\vec{u} - \text{proj}_{\vec{v}} \vec{u}$ is orthogonal to $\vec{v}$



(c) if $\vec{u} \in \text{span}\{\vec{v}\}$, then $\text{proj}_{\vec{v}} \vec{u} = \vec{u}$



(d) $\text{proj}_{\vec{v}} \vec{u} = \text{proj}_{c\vec{v}} \vec{u}$



Section 8.2 Projection and the Gram-Schmidt Process

Application Project

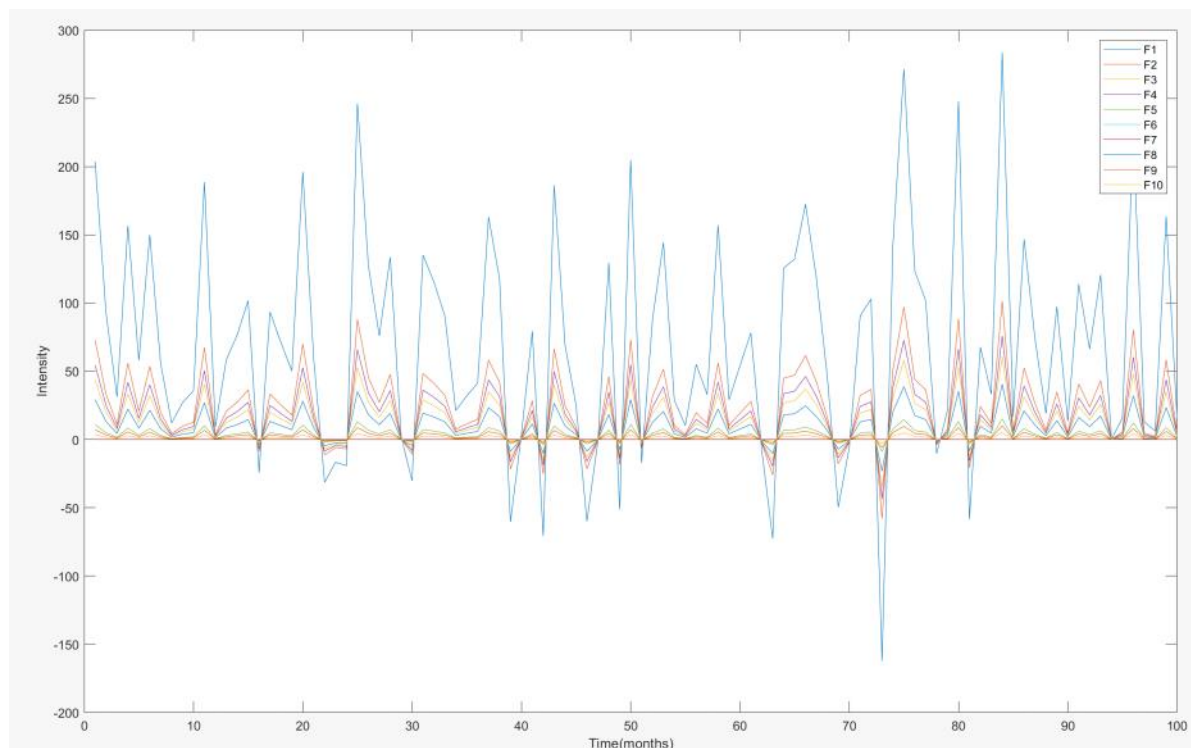
One really cool thing about light is that specific atoms emit specific frequencies of it. This means that if you could directly measure the frequencies of light coming from an extrasolar planet, you could determine its chemical composition. So you could look for water and signs of life!

You have been busy looking at a time series of light intensity data of 10 frequencies coming from the planet GND-01 which orbits the bright star AOC1337 to see if it has life. However, GND-01 is orbited by a lot of moons which have no life and light coming from them has polluted the signal from GND-01. Luckily, we know that these moons orbit the planet very quickly and so the intensity of the light at their emitted frequencies varies wildly in time while those coming from GND-01 are more steady.

Given GND-01's distance away the intensities of the 10 frequencies that would imply that it is earth like should be on the span of:

$$G_e = \begin{bmatrix} 56 \\ 20 \\ 1 \\ 15 \\ 3 \\ 12 \\ 0 \\ 8 \\ 2 \\ 12 \end{bmatrix}$$

That is if our measurements end up being a multiple of G_e we can say that GND-01 is earth like! Unfortunately when you plot each measurement taken over 100 months you see the effect of the moons interference! The measurements are all over the place! However we would expect the interference to have mean zero, meaning if we average with enough measurements it should wash out.



Section 8.2 Projection and the Gram-Schmidt Process

Your job should you choose to do it is to show for the data set GND01.mat, which you can find on Sakai in the

resources folder , is to show that the average of the data is in the span of $G_e = \begin{bmatrix} 56 \\ 20 \\ 1 \\ 15 \\ 3 \\ 12 \\ 0 \\ 8 \\ 2 \\ 12 \end{bmatrix}$.

That is I want you to show that the projection of the average measurement vector, call it $\langle GND \rangle$, onto G_e is itself which means it is on the span of G_e .

That is show $proj_{G_e} \langle GND \rangle = \langle GND \rangle$

Further I would like you to determine the multiple k such that $kG_e = \langle GND \rangle$

If you successfully do this you may hand in the Octave or MATLAB code you used with the appropriate outputs to me as a quiz replacement.

The data in GND01.mat is organized as a 10X100 matrix where the first column is the measurement of the 10 intensities on month 1 the second column month 2 and so on.

Here is what the first 14 columns looks like

Columns 1 through 14

95.5951	182.5826	84.5519	182.5040	34.1602	55.7465	46.7809	17.0834	31.6319	64.9024	221.7959	-91.0567	25.6611	24.1941
224.4501	20.1464	-197.9006	54.8143	128.1998	320.7350	52.5278	-22.9146	38.0208	34.5978	-11.5105	-44.9204	140.9477	52.4033
178.2736	-130.3384	146.5232	42.3490	-55.6998	105.6731	16.3787	-33.5374	-129.8279	-47.3140	326.2811	8.0548	-112.6420	56.7941
-5.0510	-59.2586	131.5607	-22.4514	5.4501	183.9733	-104.3385	134.4758	-77.7308	36.1255	-12.5185	130.1016	-18.3455	-72.4605
80.2324	161.4562	33.7951	-86.4252	-144.7976	169.6527	-30.0344	87.4758	21.4740	57.5705	-69.1573	125.1468	-83.5975	-32.7464
184.0004	19.3719	116.7086	40.0353	24.2107	-16.7502	185.8295	-63.4751	-20.3425	-78.8649	246.5928	60.5867	-44.3153	206.0129
-97.2107	84.6471	-10.6908	-43.8256	-38.3116	-48.7005	-30.2764	-75.0551	-206.9644	105.9693	-1.8623	120.8433	200.9424	13.8589
106.5598	-95.7845	-175.5721	147.2548	-36.8950	-158.9182	-84.3455	-167.1088	-69.6909	-159.2367	-17.8052	272.4885	-12.6414	99.6651
192.8854	-179.4753	-10.6129	-34.7377	16.4553	-95.5265	-23.5567	115.4212	28.6268	-2.6122	16.6741	67.7194	-40.9742	130.8994
115.1662	-197.7081	-55.8049	-67.5427	14.7061	-16.5797	45.6793	1.1836	119.7611	-11.2865	-88.9365	109.2136	116.3727	70.3170