

The Singular Value Decomposition (SVD)

Both diagonalization and QR factorization have wide application in applied settings in conditions when numerical error or computational cost is an issue. Another decomposition of a matrix, the singular value decomposition, is also useful and informative.

Let A be a $n \times m$ matrix with then there exist matrices

U (orthogonal $n \times n$),
 V (orthogonal $m \times m$), and

$$\Sigma = \begin{bmatrix} D_{m \times m} \\ 0_{(n-m) \times m} \end{bmatrix} \text{ if } n \geq m \text{ or } \Sigma = \begin{bmatrix} D_{m \times m} & 0_{n \times (m-n)} \end{bmatrix} \text{ if } m > n \text{ note } \Sigma \text{ is } n \times m$$

Such that $A = U \Sigma V^T$

Where $D_{ij} = \sigma_i \delta_{ij}$ and $\sigma_i = \sqrt{\lambda_i}$, which are called singular values of A and are the square roots of the positive eigen values of $A^T A$ or AA^T , which are the same. Usually ordered from greatest to least down the diagonal.

The columns of U are the orthogonal and normalized eigen vectors of AA^T which are called the left singular vectors of A ordered to be paired up with σ_i

The columns of V are the orthogonal and normalized eigen vectors of $A^T A$ which are called the right singular vectors of A ordered to be paired up with σ_i

Example: Find the singular value decomposition of $A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 0 & 2 \end{bmatrix}$ $\begin{matrix} n & m \\ 3 \times 2 \end{matrix}$

$$AA^T = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 2 & 4 \\ 2 & 4 & 0 \\ 4 & 0 & 4 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \\ 0 & 0 \end{bmatrix}$$

Find eigen values $A^T A$

$$\begin{vmatrix} 5-\lambda & 2 \\ 2 & 8-\lambda \end{vmatrix} = (5-\lambda)(8-\lambda) - 4$$

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Finding V

$$\begin{vmatrix} 5-\lambda & 2 \\ 2 & 8-\lambda \end{vmatrix} = (5-\lambda)(8-\lambda) - 4 = 0$$

$$= 40 - 8\lambda - 5\lambda + \lambda^2 - 4 = \lambda^2 - 13\lambda + 36 = 0$$

$$(\lambda - 9)(\lambda - 4) = 0$$

$$\lambda_1 = 9 \quad \lambda_2 = 4$$

$$\sigma_1 = \sqrt{9} = 3 \quad \sigma_2 = \sqrt{4} = 2$$

$$\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 5 & 2 & 4 \\ 2 & 4 & 0 \\ 4 & 0 & 4 \end{bmatrix}$$

$$A^T A = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix}$$

$$\begin{bmatrix} 5-9 & 2 \\ 2 & 8-9 \end{bmatrix}$$

$$\begin{bmatrix} 5-4 & 2 \\ 2 & 8-4 \end{bmatrix}$$

$$\begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

$$\frac{1}{2}R_1 + R_2 \rightarrow \begin{bmatrix} -4 & 2 \\ 0 & 0 \end{bmatrix}$$

$$-2R_1 + R_2 \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$$

$$\begin{cases} x_2 = t \\ -4x_1 + 2t = 0 \\ -4x_1 = -2t \\ x_1 = -\frac{1}{2}t \end{cases} \rightarrow \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} t$$

$$\begin{cases} x_2 = t \\ x_1 + 2x_2 = 0 \\ x_1 = -2t \end{cases} \rightarrow \begin{bmatrix} -2 \\ 1 \end{bmatrix} t$$

$$\vec{v}_1 = \frac{1}{\sqrt{(-1/2)^2 + 1}} \begin{bmatrix} -1/2 \\ 1 \end{bmatrix} = \frac{2}{\sqrt{5}} \begin{bmatrix} +1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} +1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}$$

$$\frac{1}{\sqrt{(-2)^2 + 1}} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

$$V = \begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix}$$

The Singular Value Decomposition (SVD)

finding U

$$AA^T = \begin{bmatrix} 5 & 2 & 4 \\ 2 & 4 & 0 \\ 4 & 0 & 4 \end{bmatrix} \rightarrow \det(A - \lambda I) = \begin{vmatrix} 5-\lambda & 2 & 4 \\ 2 & 4-\lambda & 0 \\ 4 & 0 & 4-\lambda \end{vmatrix} \quad \text{I'll expand across the bottom row}$$

$$= 4 \begin{vmatrix} 2 & 4 \\ 4-\lambda & 0 \end{vmatrix} - 0 \begin{vmatrix} 5-\lambda & 4 \\ 2 & 0 \end{vmatrix} + (4-\lambda) \begin{vmatrix} 5-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix}$$

$$= 4(2(0) - (4-\lambda)4) - 0((5-\lambda)(0) - 2(4)) + (4-\lambda)((5-\lambda)(4-\lambda) - 2(2))$$

$$= -16(4-\lambda) + (4-\lambda)(5-\lambda)(4-\lambda) - 4(4-\lambda)$$

$$= (4-\lambda)(-16 + (5-\lambda)(4-\lambda) - 4)$$

$$= (4-\lambda)(-20 + 20 - 4\lambda - 5\lambda + \lambda^2)$$

$$= (4-\lambda)(\lambda^2 - 9\lambda) = \lambda(4-\lambda)(\lambda-9)$$

$$\Rightarrow \lambda_1 = 9 \quad \lambda_2 = 4 \quad \lambda_3 = 0$$

now we find eigen vectors...

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The Singular Value Decomposition (SVD)

Finding U

$$AA^T = \begin{bmatrix} 5 & 2 & 4 \\ 2 & 4 & 0 \\ 4 & 0 & 4 \end{bmatrix}$$

$$\lambda_1 = 9, \lambda_2 = 4, \lambda_3 = 0$$

for $\lambda_1 = 9$

$$\text{Null} \left(\begin{bmatrix} 5-9 & 2 & 4 \\ 2 & 4-9 & 0 \\ 4 & 0 & 4-9 \end{bmatrix} \right)$$

$$\text{Null} \left(\begin{bmatrix} -4 & 2 & 4 \\ 2 & -5 & 0 \\ 4 & 0 & -5 \end{bmatrix} \right) \rightarrow \left[\begin{array}{ccc|c} -4 & 2 & 4 & 0 \\ 2 & -5 & 0 & 0 \\ 4 & 0 & -5 & 0 \end{array} \right]$$

$$\begin{array}{rcl} R_1 + R_3 & & -2R_2 + R_3 \\ -4 & 2 & 4 & 0 \\ +4 & 0 & -5 & 0 \\ \hline 0 & 2 & -1 & 0 \end{array} \quad \begin{array}{rcl} -4 & 10 & 0 & 0 \\ +4 & 0 & -5 & 0 \\ \hline 0 & 10 & -5 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} -4 & 2 & 4 & 0 \\ 0 & 10 & -5 & 0 \\ 0 & 2 & -1 & 0 \end{array} \right]$$

$-5R_3 + R_2$

$$\begin{array}{rcl} 0 & -10 & +5 & 0 \\ 0 & 10 & -5 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} -4 & 2 & 4 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 \end{array} \right] \rightarrow \left[\begin{array}{ccc|c} -4 & 2 & 4 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \rightarrow \begin{cases} -4x_1 + 2x_2 + 4x_3 = 0 \\ 2x_2 - x_3 = 0 \end{cases}$$

$$2x_2 - t = 0 \Rightarrow x_2 = \frac{1}{2}t$$

$$-4x_1 + 2\left(\frac{1}{2}t\right) + 4t = 0$$

$$-4x_1 + 5t = 0$$

$$\Rightarrow x_1 = \frac{5}{4}t$$

$$\Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5/4 \\ 1/2 \\ 1 \end{bmatrix} t$$

$$\text{Normalize } \frac{1}{\sqrt{(5/4)^2 + (1/2)^2 + 1^2}} \begin{bmatrix} 5/4 \\ 1/2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{25/16 + 4/16 + 16/16}} \begin{bmatrix} 5/4 \\ 1/2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{45}} \begin{bmatrix} 5/4 \\ 1/2 \\ 1 \end{bmatrix} = \frac{4}{\sqrt{45}} \begin{bmatrix} 5/4 \\ 1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} 5/\sqrt{45} \\ 2/\sqrt{45} \\ 4/\sqrt{45} \end{bmatrix}$$

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for $\lambda_2 = 4$

$$\text{Null} \left(\begin{bmatrix} 5-4 & 2 & 4 \\ 2 & 4-4 & 0 \\ 4 & 0 & 4-4 \end{bmatrix} \right)$$

$$\text{Null} \left(\begin{bmatrix} 1 & 2 & 4 \\ 2 & 0 & 0 \\ 4 & 0 & 0 \end{bmatrix} \right) \rightarrow \left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 2 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 \end{array} \right]$$

$$\begin{array}{rcl} -2R_2 + R_1 & & -4R_3 + R_1 \\ 1 & 2 & 4 & 0 \\ -2 & 0 & 0 & 0 \\ \hline 1 & 2 & 4 & 0 \end{array}$$

$$x_1 + 2x_2 + 4x_3 = 0$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 4 & 0 \\ 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \rightarrow \begin{cases} 2x_1 = 0 \Rightarrow x_1 = 0 \\ x_3 = t \end{cases}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -2t \\ t \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

$$\text{Normalize } \frac{1}{\sqrt{(-2)^2 + 1^2}} \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ -2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$

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for $\lambda_3 = 0$ Null $\left(\begin{bmatrix} 5 & 2 & 4 \\ 2 & 4 & 0 \\ 4 & 0 & 4 \end{bmatrix} \right)$

$$\left[\begin{array}{ccc|c} 5 & 2 & 4 & 0 \\ 2 & 4 & 0 & 0 \\ 4 & 0 & 4 & 0 \end{array} \right] \quad \begin{array}{l} -2R_2 + R_3 \\ -4R_1 + 5R_2 \\ 4R_1 + 5R_2 \end{array}$$

$$\begin{array}{ccc|c} -4 & -8 & 0 & 0 \\ 4 & 0 & 4 & 0 \\ \hline 0 & -8 & 4 & 0 \end{array} \quad \begin{array}{ccc|c} -20 & -8 & -16 & 0 \\ 20 & 0 & 20 & 0 \\ \hline 0 & -8 & 4 & 0 \end{array}$$

$$\left[\begin{array}{ccc|c} 5 & 2 & 4 & 0 \\ 0 & -8 & 4 & 0 \\ 0 & -8 & 4 & 0 \end{array} \right] \quad R_2 + R_3$$

$$\left[\begin{array}{ccc|c} 5 & 2 & 4 & 0 \\ 0 & -8 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right] \quad \begin{array}{l} 5x_1 + 2x_2 + 4x_3 = 0 \\ -8x_2 + 4x_3 = 0 \\ \Rightarrow x_3 = t \end{array} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ 1/2 t \\ t \end{bmatrix} = \begin{bmatrix} -1 \\ 1/2 \\ 1 \end{bmatrix} t$$

Normalizing $\frac{1}{\sqrt{(-1)^2 + (1/2)^2 + (1)^2}} = \frac{1}{\sqrt{1 + 1/4 + 1}} = \frac{1}{\sqrt{2 1/4 + 1/4 + 1/4}} = \frac{1}{\sqrt{9/4}} = 2/3$

$$= 2/3 \quad \text{so} \quad \frac{2}{3} \begin{bmatrix} -1 \\ 1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$$

So we have

$$\vec{U}_1 = \begin{bmatrix} 5/\sqrt{45} \\ 2/\sqrt{45} \\ 4/\sqrt{45} \end{bmatrix} \quad \vec{U}_2 = \begin{bmatrix} 0 \\ -2/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix} \quad \vec{U}_3 = \begin{bmatrix} -2/3 \\ 1/3 \\ 2/3 \end{bmatrix}$$

so

$$U = \begin{bmatrix} 5/\sqrt{45} & 0 & -2/3 \\ 2/\sqrt{45} & -2/\sqrt{3} & 1/3 \\ 4/\sqrt{45} & 1/\sqrt{3} & 2/3 \end{bmatrix}$$

$$V = \begin{bmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix} \quad \Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$$

The Singular Value Decomposition (SVD)

octave:71> A=[1 2; 2 0; 0 2]
A =

```
1 2
2 0
0 2
```

octave:72> [U1, DU1]=eig(A*A')
U1 =

```
2/3 0 963/1292
-1/3 -305/341 963/3230
-2/3 1292/2889 963/1615
```

DU1 =

Diagonal Matrix

```
0 0 0
0 4 0
0 0 9
```

octave:73> Sigma=[3 0; 0 2; 0 0]
Sigma =

```
[3 0]
[0 2]
[0 0] = Σ
```

octave:74> U=[U1(:,3) U1(:,2) U1(:,1)]
U =

```
963/1292 0 2/3
963/3230 -305/341 -1/3
963/1615 1292/2889 -2/3
```

octave:75> [V1, DV1]=eig(A'*A)
V1 =

```
[-305/341 1292/2889]
[1292/2889 5473/6119]
```

DV1 =

Diagonal Matrix

```
[4 0]
[0 9]
```

octave:76> V=[V1(:,2) V1(:,1)]
V =

```
1292/2889 -305/341
5473/6119 1292/2889
```

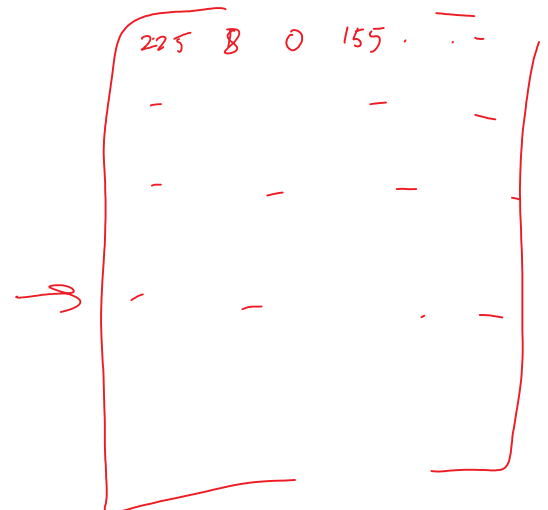
octave:77> U*Sigma*V'
ans =

```
[1 2]
[2 0]
[0 2]
```

$$\begin{matrix} \downarrow & \downarrow & \downarrow \\ [U, S, V] = \text{svd}(A) \end{matrix}$$

The Singular Value Decomposition (SVD)

Applications of the SVD



562 x 562

Image Compression: We can think of the image above as a matrix where we assign a gray scale intensity to each pixel position. So we have a matrix of numbers between 0 and 255.

If you recall, in the matrix Σ is ordered with the largest singular values first and going down from there. Our SVD product can be written in the form

$$A = U \Sigma V^T = \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T + \dots + \sigma_n \vec{u}_n \vec{v}_n^T$$

Handwritten notes:

- $U = n \times n$, $\vec{v}_i = n \times 1$
- $V = m \times m$, $\vec{v}_i = m \times 1$
- $\vec{v}_i^T = 1 \times m$
- $\vec{u}_i$ is $n \times 1$, $\vec{v}_i^T$ is $1 \times m$ so $\vec{u}_i \vec{v}_i^T$ is $n \times m$ the same as A
- $n \times 1 \times 1 \times m = n \times m$
- 0.000002

$$\sigma_1 > \sigma_2 > \sigma_3$$

$$\sigma_n = 0.000002$$

From this we might note that if the smallest singular values are very small, they don't contribute much information to A . So what would happen to the image if we throw away some of terms being scaled by small singular values?

$$A \approx \sigma_1 \vec{u}_1 \vec{v}_1^T + \sigma_2 \vec{u}_2 \vec{v}_2^T + \dots + \sigma_k \vec{u}_k \vec{v}_k^T$$

Where $k < n$ keeping in mind that $\sigma_1 > \sigma_2 > \sigma_3 > \dots > \sigma_n$

In matlab matrix multiplication notation keeping the first 10 terms would read

$$U(:,1:10)S(1:10,1:10)V(:,1:10)'$$

Transposed

The Singular Value Decomposition (SVD)



Keeping only 10 singular values

$U(:,1:10)S(1:10,1:10)V(:,1:10)'$



Keeping only 50 singular values

$U(:,1:50)S(1:50,1:50)V(:,1:50)'$



Keeping only 100 singular values

$U(:,1:100)S(1:100,1:100)V(:,1:100)'$



Original 562X562

The Singular Value Decomposition (SVD)

The condition number of a matrix

The condition number of a matrix A is $\kappa(A) = \frac{\sigma_{\max}}{\sigma_{\min}}$ This number gives a sense of how terrible an inversion of a matrix might be!

```
>> B=randi([-10 10], 5, 3)
```

B =

-2	-4	-8
1	-1	-3
-2	-10	-6
3	10	0
3	-7	-3

```
>> A=[B B(:,2)-0.001, B(:,1)-.01]
```

A =

-2	-4	-8	-4001/1000	-201/100
1	-1	-3	-1001/1000	99/100
-2	-10	-6	-10001/1000	-201/100
3	10	0	9999/1000	299/100
3	-7	-3	-7001/1000	299/100

```
>> [U,S,V]=svd(A)
```

U =

-431/1274	-759/1090	385/1053	569/1102	302/15201
-427/4964	-421/1285	-229/1069	-466/1487	-588/683
-1139/1791	-231/1943	239/1902	-268/395	306/943
1171/2076	-3543/5656	-1316/4877	-339/1138	604/1689
-941/2386	-356/10045	-2995/3501	547/1864	305/1919

S =

8644/353	0	0	0	0
0	4762/609	0	0	0
0	0	2905/442	0	0
0	0	0	59/17298	0
0	0	0	0	*

V =

295/3046	-243/2783	-997/1433	-272/387	-221/3141
563/851	-415/1893	269/2252	-538/7655	489/695
1107/3404	2049/2173	-511/7148	-30/19169	*
698/1055	-966/4411	295/2467	188/2669	-489/695
265/2726	-1028/12093	-2441/3515	2785/3954	221/3141

```
>> svd(A)
```

ans =

8644/353
4762/609
2905/442
59/17298
1/5614388000771653

This condition number is huge!
We would expect any inversion to be horrible!

```
>> S(1,1)/S(5,5)
```

ans =

137480924864676784

$\kappa(A) = 5614388000771653$

The Singular Value Decomposition (SVD)

```
>> x=[1;2;3;4;5]
```

```
x =
```

```
1
2
3
4
5
```

```
>> b=A*x
```

```
b =
```

```
-60.0540
-9.0540
-90.0540
77.9460
-33.0540
```

$$x = A^{-1}b$$

```
>> inv(A)*b
```

Warning: Matrix is close to singular or badly scaled. Results may be inaccurate. RCOND = 3.663224e-18.

```
ans =
```

```
1.0396
3.5995
3.0000
12.4005
3.4275
```

$$x = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

Why do we have this problem? Let's think about inverting the SVD version of A...

$$A\vec{x} = \vec{b}$$

$$\downarrow$$

$$\vec{x} = A^{-1}\vec{b}$$

$$(USV^T)\vec{x} = \vec{b}$$

$$\downarrow$$

$$USV^T\vec{x} = \vec{b} \quad U^T = U^{-1}$$

$$I - U^T U S V^T \vec{x} = U^T \vec{b}$$

$$S V^T \vec{x} = U^T \vec{b}$$

$$S^{-1} S V^T \vec{x} = S^{-1} U^T \vec{b}$$

$$I - V^T \vec{x} = S^{-1} (U^T \vec{b}) \rightarrow \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

$$S^{-1} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} 1/\sigma_1 & & \\ & \ddots & \\ & & 1/\sigma_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$$\begin{bmatrix} 1/\sigma_1 y_1 \\ \vdots \\ 1/\sigma_n y_n \end{bmatrix} \rightarrow \underline{\underline{1/\sigma_n y_n}} \text{ numerical error}$$

$$S = \begin{bmatrix} \sigma_1 & \sigma_2 & 0 \\ 0 & \ddots & \sigma_n \\ 0 & 0 & \ddots \end{bmatrix}$$

$$S^{-1} = \begin{bmatrix} 1/\sigma_1 & & \\ & 1/\sigma_2 & \\ & & \ddots \\ & & & 1/\sigma_n \end{bmatrix}$$

House