

Section 9.1 Linear Transformations

Let V, W be vector spaces. Then $T: V \rightarrow W$ is a linear transformation if for every $\vec{v}_1, \vec{v}_2 \in V$

(a) $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$

(b) $T(c\vec{v}_1) = cT(\vec{v}_1)$

$\mathbb{R}^n \quad \mathcal{O}_{\mathbb{R}^n} = \begin{bmatrix} 0 & & \\ & \ddots & \\ & & 0 \end{bmatrix} \quad P^n \quad \mathcal{O}_{P^n} = 0$

$c \in \mathbb{R}$

This is pretty much the same definition we had in section 3.1, however now we allow for general vector spaces!

Let $T: V \rightarrow W$ be a linear transformation $T(\vec{v}) = \vec{w}$.

We call the set of input vectors $\{\vec{v}\} \in V$ the **Domain** of T

We call the set of output vectors $\{\vec{w}\} \in W$ the **Codomain** of T

For a specific vector $\vec{v}$ in the domain of T we call $\vec{w} = T(\vec{v})$ the **Image** of $\vec{v}$ under T

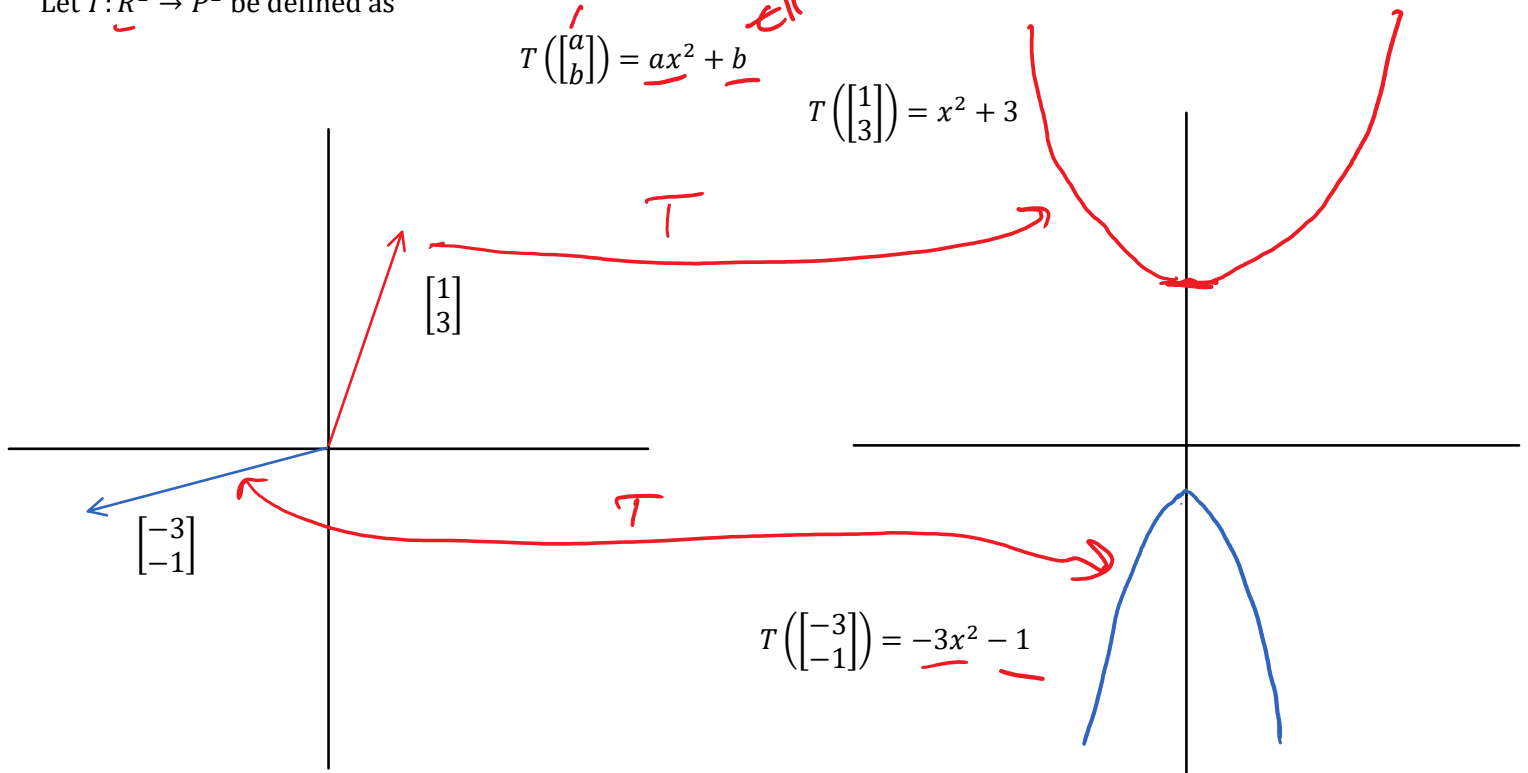
The collection of the **images** of all of the vectors in the domain is called the **Range** of T ($\text{range}(T)$)

The set of all input vectors $\{\vec{v}\} \in V$ such that $T(\vec{v}) = \vec{0}$ is called the **Kernel** of T ($\ker(T)$)

Let $T: \mathbb{R}^2 \rightarrow P^2$ be defined as

$T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = ax^2 + b$

$T\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}\right) = x^2 + 3$



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Let $T: \mathbb{R}^2 \rightarrow \mathbb{P}^2$ be defined as $T\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = ax^2 + b$

Is it linear?

$$\vec{v}_1 = \begin{bmatrix} a_1 \\ b_1 \end{bmatrix} \quad \vec{v}_2 = \begin{bmatrix} a_2 \\ b_2 \end{bmatrix} \quad \vec{v}_1 + \vec{v}_2 = \begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \end{bmatrix}$$

(a) $T(\vec{v}_1 + \vec{v}_2) = T(\vec{v}_1) + T(\vec{v}_2)$

$$T(\vec{v}_1 + \vec{v}_2) = T\left(\begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \end{bmatrix}\right) = (a_1 + a_2)x^2 + (b_1 + b_2) \leftarrow \text{Same}$$

$$T(\vec{v}_1) + T(\vec{v}_2) = T\left(\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}\right) + T\left(\begin{bmatrix} a_2 \\ b_2 \end{bmatrix}\right) = a_1x^2 + b_1 + a_2x^2 + b_2 \\ = a_1x^2 + a_2x^2 + b_1 + b_2 \\ = (a_1 + a_2)x^2 + (b_1 + b_2)$$

(b) $T(c\vec{v}_1) = cT(\vec{v}_1)$

$c \in \mathbb{R}$

$$c\begin{bmatrix} a_1 \\ b_1 \end{bmatrix} = \begin{bmatrix} ca_1 \\ cb_1 \end{bmatrix}$$

$$T(c\vec{v}_1) = T\left(\begin{bmatrix} ca_1 \\ cb_1 \end{bmatrix}\right) = ca_1x^2 + cb_1 = c(a_1x^2 + b_1) \leftarrow cT\left(\begin{bmatrix} a_1 \\ b_1 \end{bmatrix}\right)$$

Let $T: V \rightarrow W$ be a linear transformation then $\ker(T)$ is a subspace of V and if S is a subspace of V then $T(S)$ is a subspace of W

Proof:

assume S is a subspace of V show $T(S)$ is a subspace of W

Is $0_W \in T(S)$ $0_V \in S$ $T(0_V) = 0_W$

If $\vec{w}_1, \vec{w}_2 \in T(S)$ does $\vec{w}_1 + \vec{w}_2 \in T(S)$?

Let $\vec{w}_1, \vec{w}_2 \in T(S)$ then there are $\vec{v}_1, \vec{v}_2 \in S$ s.t. $T(\vec{v}_1) = \vec{w}_1$ $T(\vec{v}_2) = \vec{w}_2$

$\vec{w}_1 + \vec{w}_2 = T(\vec{v}_1) + T(\vec{v}_2) = T(\vec{v}_1 + \vec{v}_2) \in T(S)$ because $\vec{v}_1 + \vec{v}_2 \in S$

If $\vec{w}_1 \in T(S)$ does $c\vec{w}_1 \in T(S)$?

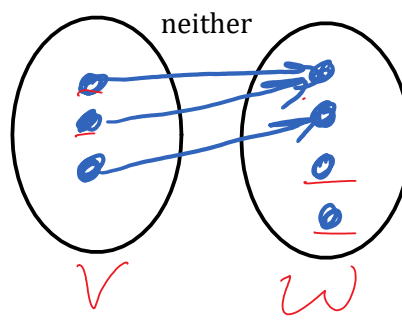
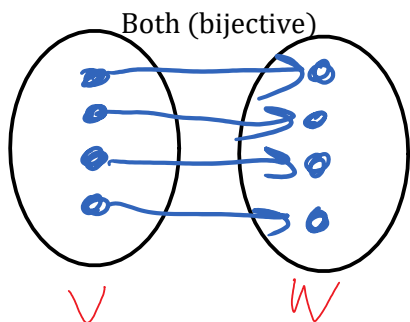
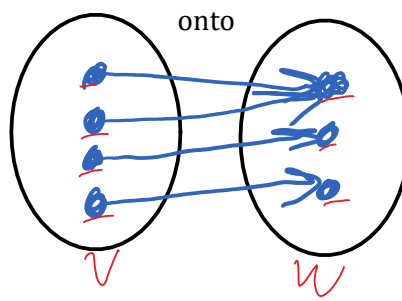
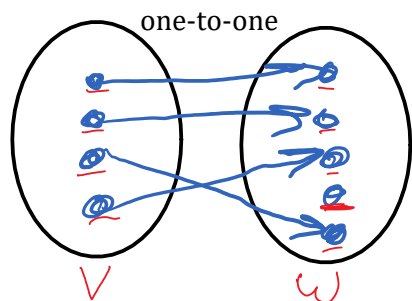
$c\vec{w}_1 = cT(\vec{v}_1) = T(c\vec{v}_1) \in T(S)$ because $\vec{v}_1 \in S \Rightarrow c\vec{v}_1 \in S$

Section 9.1 Linear Transformations

Let $T: V \rightarrow W$ be a linear transformation then

(a) T is one-to-one (injective) if for every vector $\vec{w} \in W$ there exists at most one vector $\vec{v} \in V$ such that $T(\vec{v}) = \vec{w}$.

(b) T is onto (surjective) if for every vector $\vec{w} \in W$ there exists at least one vector $\vec{v} \in V$ such that $T(\vec{v}) = \vec{w}$.



Let $T: \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}$ be defined as

$$T(A) = T\left(\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\right) = \text{tr}(A) = a_{11} + a_{22}$$

Trace of A (Sum of the diagonal elements of A)

Is T onto?

Let $b \in \mathbb{R}$ then $T\left(\begin{bmatrix} 0 & a_{12} \\ a_{21} & b \end{bmatrix}\right) = 0 + b = b$

Is T 1-1?

Let $b \in \mathbb{R}$ then $T\left(\begin{bmatrix} 0 & a_{12} \\ a_{21} & b \end{bmatrix}\right) = 0 + b = b$

$T\left(\begin{bmatrix} b & a_{12} \\ a_{21} & 0 \end{bmatrix}\right) = b + 0 = b$

NO!

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Let $T: V \rightarrow W$ be a linear transformation, then T is 1-1 if and only if $T(\vec{v}) = \vec{0}_W$ only has the trivial solution of $\vec{v} = \vec{0}_V$.

Suppose $T(\vec{v}) = \vec{0}_W$ has the trivial solution and that T is not 1-1. Then there should exist $\vec{v}_1, \vec{v}_2$ such that $T(\vec{v}_1) = T(\vec{v}_2) = \vec{w}$

$$\vec{w} = \vec{w} = \vec{0}_W$$

$$T(\vec{v}_1) - T(\vec{v}_2) = \vec{0}_W$$

And because T is linear...

$$T(\vec{v}_1) + T(-\vec{v}_2) = \vec{0}_W$$

$$\vec{v} = \vec{v}_1 - \vec{v}_2 \in V$$

$$T(\vec{v}) = \vec{0}_W \quad \text{since only Trivial soln}$$

$$T(\vec{v}_1 - \vec{v}_2) = \vec{0}_W$$

$$\vec{v}_1 - \vec{v}_2 = \vec{0}_V$$

$$\vec{v}_1 = \vec{v}_2$$

Suppose T is 1-1

Suppose there is some $\vec{v}_1 \neq \vec{0}$ such that $T(\vec{v}_1) = \vec{0}_W$

But $0 \cdot T(\vec{v}_1) = 0 \cdot \vec{0}_W$ and we proved in section 7.1 that...

$$0 \cdot T(\vec{v}_1) = \vec{0}_W$$

$$T(\vec{v}_1) = \vec{0}_W$$

$$T(0 \cdot \vec{v}_1) = \vec{0}_W$$

$$T(\vec{0}_V) = \vec{0}_W$$

since T is 1-1

$$T(\vec{0}_V) = \vec{0}_W$$

$$\vec{v}_1 = \vec{0}_V$$

Let V be a vector space and suppose that $\vec{v} \in V$. Then,

- (a) $\vec{0}$ is a vector in V then $\vec{v} + \vec{0} = \vec{v}$
- (b) if $-\vec{v}$ is the additive inverse of $\vec{v}$ then $\vec{v} + (-\vec{v}) = \vec{0}$
- (c) the additive inverse of $\vec{v}$ is unique
- (d) $0 \cdot \vec{v} = \vec{0}$
- (e) $-1 \cdot \vec{v} = -\vec{v}$

Section 9.1 Linear Transformations

Let $T: V \rightarrow W$ be a linear transformation. Suppose that $S_v = \{\vec{v}_1, \dots, \vec{v}_k\}$ is a subset of V and $S_w = \{\vec{w}_1, \dots, \vec{w}_k\}$ as subset of W with $T(\vec{v}_i) = \vec{w}_i$ for *each* vector in S_v . Then if S_w is linearly independent, so is S_v .

Proof.

Start with $c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_k \vec{v}_k = 0_V$. If the $\vec{v}$'s are to be linearly independent then the only solution to the equation should be all of the c 's being zero.

$$\rightarrow c_1 = c_2 = \dots = c_k = 0$$

Now we assume the $\vec{w}$'s are linearly independent and we examine keeping in mind that T is linear

$$T(c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_k \vec{v}_k) = T(0_V)$$

$$T(c_1 \vec{v}_1) + T(c_2 \vec{v}_2) + \dots + T(c_k \vec{v}_k) = 0_W$$

$$c_1 T(\vec{v}_1) + c_2 T(\vec{v}_2) + \dots + c_k T(\vec{v}_k) = 0_W$$

$$c_1 \vec{w}_1 + c_2 \vec{w}_2 + \dots + c_k \vec{w}_k = 0_W$$

$$c_1 = c_2 = \dots = c_k = 0$$

Prove that $\{\sin(x), \cos(x)\}$ on $C(R)$ are linearly independent by finding a linear transformation from them to the linearly independent vectors $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ in R^2 .

$$T(\sin(x)) = \begin{bmatrix} \sin(\pi/2) \\ \sin(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad T(f) = \begin{bmatrix} f(\pi/2) \\ f(0) \end{bmatrix}$$

$$T(\cos(x)) = \begin{bmatrix} \cos(\pi/2) \\ \cos(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\Rightarrow \sin(x), \cos(x)$ are L.I.

$$c_1 \sin(x) + c_2 \cos(x) = 0 \quad c_1 = c_2 = 0$$

$$\begin{aligned} T(f+g) &= \begin{bmatrix} f(\pi/2) + g(\pi/2) \\ f(0) + g(0) \end{bmatrix} \\ T(f) + T(g) &= \begin{bmatrix} f(\pi/2) \\ f(0) \end{bmatrix} + \begin{bmatrix} g(\pi/2) \\ g(0) \end{bmatrix} \\ T(cf) &= \begin{bmatrix} cf(\pi/2) \\ cf(0) \end{bmatrix} = c \begin{bmatrix} f(\pi/2) \\ f(0) \end{bmatrix} = c T(f) \end{aligned}$$

T is linear

Section 9.1 Linear Transformations

Let $T: V \rightarrow W$ be a linear transformation with V and W finite dimensional. Then,

$$\dim(V) = \dim(\ker(T)) + \dim(\text{range}(T))$$

Proof:

$$T(\vec{v}_1) = T(\vec{v}_2) = T(\vec{v}_3) = \dots = T(\vec{v}_k) = \vec{0}_W$$

Let $\{\vec{v}_1, \dots, \vec{v}_k\}$ be a basis for $\ker(T)$ so that $\dim(\ker(T)) = k$ now extend the list with $\{\vec{v}_{k+1}, \dots, \vec{v}_m\}$ to obtain a basis for V of

$$\{\vec{v}_1, \dots, \vec{v}_k, \vec{v}_{k+1}, \dots, \vec{v}_m\}$$

So here $\dim(V) = m$

Now take $\vec{v}$ in V using our basis it can be written in a unique way as,

$$\vec{v} = c_1 \vec{v}_1 + c_2 \vec{v}_2 + \dots + c_k \vec{v}_k + c_{k+1} \vec{v}_{k+1} + \dots + c_m \vec{v}_m$$

Now we examine $T(\vec{v})$ keeping in mind that $\vec{v}_1 \dots \vec{v}_k$ are in $\ker(T)$. Our end goal will be to show that the linear transformation of all of the $\{\vec{v}_{k+1}, \dots, \vec{v}_m\}$ form a basis for $\text{range}(T)$ if so then then it is $m - k$ dimensional and we will have our theorem!

$$T(\vec{v}) = T(c_1 \vec{v}_1 + \dots + c_k \vec{v}_k + c_{k+1} \vec{v}_{k+1} + \dots + c_m \vec{v}_m)$$

$$= \underbrace{c_1 T(\vec{v}_1) + \dots + c_k T(\vec{v}_k)}_{\vec{0}} + \underbrace{c_{k+1} T(\vec{v}_{k+1}) + \dots + c_m T(\vec{v}_m)}$$

$$\text{Span}\{\vec{v}_{k+1}, \dots, \vec{v}_m\} = \text{Range}(T) \Rightarrow \dim(\text{Range}(T)) = m - k$$

$$\dim(V) = m$$

$$\dim(\ker(T)) = k \quad \dim(\text{range}(T)) = m - k$$

$$k + m - k = m = \dim(V)$$

Section 9.1 Linear Transformations

Let $T: V \rightarrow W$ be a linear transformation with V and W finite dimensional then,

(a) If T is onto, then $\dim(V) \geq \dim(W)$

(b) If T is one-one then $\dim(V) \leq \dim(W)$

Proof:

If T is onto $W = \text{range}(T)$ so $\dim(W) = \dim(\text{range}(T))$

So $\dim(V) = \dim(\ker(T)) + \dim(\text{range}(T))$ so if there is any thing in the kernel we add that dimension to that of W and (a) is true.

$\dim(\text{range}(T))$

If T is 1-1 then $\dim(\ker(T)) = 0$

So $\dim(V) = 0 + \dim(\text{range}(T))$ and either $\text{range}(T) = W$ or it is not onto and $\text{range}(T) \subset W$ and so (b) is true.

Section 9.1 Linear Transformations

$$A = [\vec{a}_1 \ \vec{a}_2 \ \dots \ \vec{a}_k] \quad \text{6-S}$$

$$Q = [\vec{q}_1 \ \vec{q}_2 \ \dots \ \vec{q}_k]$$

$\vec{q}_k$ Linearly dependent

$\{\vec{a}_1, \dots, \vec{a}_{k-1}\}$ basis $\text{col}(A)$

$$\vec{s} \in \text{Span}(\vec{q}_1, \dots, \vec{q}_k)$$

$$\vec{s} = \text{proj}_{\vec{q}_1}(\vec{s})\vec{q}_1 + \text{proj}_{\vec{q}_2}(\vec{s})\vec{q}_2 + \dots \quad \vec{q}_k \in \text{Span}\{\vec{a}_1, \dots, \vec{a}_{k-1}\}$$

$$\vec{q}_k \in \text{Span}\{\vec{q}_1, \dots, \vec{q}_{k-1}\} \quad \text{basis } \text{col}(A)$$

$$\vec{q}_k = \vec{a}_k - \text{proj}_{\vec{q}_{k-1}}(\vec{a}_k)\vec{q}_{k-1} - \text{proj}_{\vec{q}_{k-2}}(\vec{a}_k)\vec{q}_{k-2} - \dots - \text{proj}_{\vec{q}_1}(\vec{a}_k)\vec{q}_1$$

$$\vec{a}_k - \underbrace{(\text{proj}_{\vec{q}_{k-1}}(\vec{a}_k) + \dots + \text{proj}_{\vec{q}_1}(\vec{a}_k))}_{\vec{a}_k}$$

$$\vec{q}_k = \vec{a}_k - \vec{a}_k = \vec{0}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$R = \begin{bmatrix} \vec{a}_i \cdot \vec{a}_j \\ 0 \end{bmatrix}$$

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$$Q = [\vec{q}_1 \dots \vec{q}_{k-1}] \quad \underline{QR = A}$$

$$\text{col}(A) = \text{Span} \{ \underline{\vec{q}_1 \dots \vec{q}_{k-1}} \}$$

$$\underline{(\text{col}(A))^\perp = \text{Null}(A^T)}$$