

Extra Credit 2

Math 1050 Summer 2013

For this extra credit you will learn something useful if you have any student loans out. For this exercise we will assume that you have two student loans out one for \$7000 with an interest rate of 6.8 % and the other for \$5000 with an interest rate of 7.8% . Lets say they are both Federal subsidized loans but as happened recently in congress the rates went up. So you have one loan at a lower rate 6.8% and after the rate increase you had to take out another at 7.8%. For most student loans you don't pay interest until you graduate. The smart thing to do would be to save money while in school and pay off as much as you can when you graduate but that can be difficult. Lets assume that you don't save any money while in school and then after you graduate you get a job that allows you to budget \$300 per month to pay off your loans. The minimum on both is \$50 but you know that you can save money in the long run if you pay off the loans as quickly as possible. The question then becomes how much of your \$300 should go to the \$7000 loan at 6.8% and how much should go to the \$5000 loan at 7.8% to minimize interest and pay off time? That is what we will figure out.

Most student loans are calculated using daily simple interest. That is to say the interest charged to you is the number of days since your last payment divided by 365 (number of days in a year) multiplied by your interest rate times your loan balance. In an equation:

$$I = \frac{d}{365} rP$$

Where (I) is the interest charged d the number of days since your last payment (r) the interest rate and (P) the current loan balance or principle (what you still owe). For this we will assume that you intend to make a payment at the end of each month and that every month has 31 days to get a bit of an over estimate. That means that say the first month you make a payment on the \$7000 your interest would be

$$I = \frac{31}{365} (.068) \$7000 = \$40.42$$

The payment you make that month is split between paying off your interest from that month and what you owe. So let's say you make a payment of \$100 then \$40.42 goes to paying the interest and \$100-\$40.42 = \$59.58 goes to paying off the principal so after your first payment you will have a loan balance of \$7000-\$59.58 = \$6940.40 The next month the process repeats except the interest is calculated using \$6940.40 instead of 7000 dollars so you the second month your interest would be

$$I = \frac{31}{365} (.068) \$6940.40 = \$40.08$$

Then you split the \$100 dollar payment again and the process repeats until you don't owe any more money. Notice that after 2 months you have already paid almost \$80 in interest. If instead of 100 dollars you had made a payment of \$200 the first month then the payment to principle would have been $\$200 - \$40.42 = \$159.58$ your principle the second month would then be $\$7000 - \$159.58 = \$6840.40$. Then the interest for the second month would instead be

$$I = \frac{31}{365} (.068) \$6840.40 = \$39.50$$

While not seemingly a big difference over each month paying less interest saves you money and further more note that your principle is 100 dollars less this means you get to the point where you owe nothing in less time. Then you can spend that 300 dollars a month on something other then paying down debt.

Ok now that you probably get how this works lets do this in general terms so we can optimize our payments for both loans.

Since we will assume each month has 31 days for simplicity we will define the interest factor

$$f = \frac{31}{365} r$$

Where r is the interest rate for the loan. Let P_0 be the amount you took out and P_m you monthly payment. Through the power of mathamagic you can calculate that the principle balance (what you still owe) for the n th month is given by:

$$P_n = P_0(1 + f)^n - \frac{P_m}{f} ((1 + f)^n - 1)$$

So here if you plug in 2 for n you get what you owe after the second month if you plug in $n=30$ you get what you still owe at the end of the 30th month. One important question you might ask is how long will it take to pay off a loan for a given monthly payment P_m ? Well we can use the above equation to solve that problem. When the loan is paid off we owe nothing so $P_n = 0$. This means we can plug in 0 for P_n and then solve for n the number of months. We would get this equation: that we can solve for n .

$$0 = P_0(1 + f)^n - \frac{P_m}{f} ((1 + f)^n - 1)$$

$$P_0(1 + f)^n = \frac{P_m}{f} ((1 + f)^n - 1)$$

$$P_0(1+f)^n = \frac{P_m}{f}(1+f)^n - \frac{P_m}{f}$$

$$\frac{P_m}{f}(1+f)^n - P_0(1+f)^n = \frac{P_m}{f}$$

$$\left(\frac{P_m}{f} - P_0\right)(1+f)^n = \frac{P_m}{f}$$

$$(1+f)^n = \frac{P_m}{f\left(\frac{P_m}{f} - P_0\right)}$$

$$(1+f)^n = \frac{P_m}{(P_m - fP_0)}$$

$$\ln((1+f)^n) = \ln\left(\frac{P_m}{(P_m - fP_0)}\right)$$

$$n \ln(1+f) = \ln\left(\frac{P_m}{(P_m - fP_0)}\right)$$

$$n = \frac{\ln\left(\frac{P_m}{(P_m - fP_0)}\right)}{\ln(1+f)}$$

This gives us a formula for the number of months n it will take to pay off the loan with initial principle P_0 interest rate factor f and monthly payment P_m . Once we know how many months it will take to pay off a loan we could calculate the total interest paid using this mathamagical formula:

$$I = P_0[(1+f)^{n+1} - 1] - \frac{P_m}{f}[(1+f)^{n+1} - 1 - nf]$$

You will actually have all of necessary tools to derive each of these formulas by the end of our class but if you want an extra 5 points on this extra credit you can derive the above equation using our formula for P_n (what you still owe in the n th month) and the formulas for geometric series in chapter 9.

Ok now that we have all of the equations we need lets get to optimizing our payments.

We have a total budget of \$300 dollars a month to throw at the student loans and we want to split our payments between the loans to minimize both the pay off time

and the interest we pay. Splitting the payments equally does not make sense as the interest rates and amount owed are both different. We might think to throw more at the loan with a higher interest rate but we owe less on that one so the monthly interest is a higher percentage of a lower number. Also Even though we owe more on the lower interest rate the lower interest rate means we might pay less interest if we throw more money at that loan. It sounds like a headache to decide so we will use math to help us.

One thing is clear if we minimize the total time it takes us to pay off the loan we will also minimize the interest we pay. So what we will do is write an equation that represents the time it takes us to pay off both loans each with their own monthly payment and find which payments minimize that time. We know that the time to pay off the loan is given by:

$$n = \frac{\ln\left(\frac{P_m}{(P_m - fP_0)}\right)}{\ln(1 + f)}$$

Lets say P_m^1 is the monthly payment for the \$7000 loan at 6.8% interest and P_m^2 the monthly payment for the for the \$5000 loan at %7.8 interest. Note the super script 2 here is not squaring just standing for second monthly payment. Then the time needed to pay off each loan given the payments would be:

$$n_1 = \frac{\ln\left(\frac{P_m^1}{\left(P_m^1 - \frac{31}{365}(.068)(7000)\right)}\right)}{\ln\left(1 + \frac{31}{365}(.068)\right)}$$

and

$$n_2 = \frac{\ln\left(\frac{P_m^2}{\left(P_m^2 - \frac{31}{365}(.078)(5000)\right)}\right)}{\ln\left(1 + \frac{31}{365}(.078)\right)}$$

Now we want to minimize both at the same time and one way we can do this is by minimizing the sum of them $n_1 + n_2$ So we will minimize

$$n_1 + n_2 = \frac{\ln\left(\frac{P_m^1}{\left(P_m^1 - \frac{31}{365}(.068)(7000)\right)}\right)}{\ln\left(1 + \frac{31}{365}(.068)\right)} + \frac{\ln\left(\frac{P_m^2}{\left(P_m^2 - \frac{31}{365}(.078)(5000)\right)}\right)}{\ln\left(1 + \frac{31}{365}(.078)\right)}$$

Now we need to get everything in terms of just one variable instead of 2. We can do this since we know that we only have \$300 to throw at the loans each month so this means that the sum of the payments to each loan has to be \$300

$$P_m^1 + P_m^2 = 300$$

Which means that:

$$P_m^1 = 300 - P_m^2$$

So in the equation for $n_1 + n_2$ above that we want to minimize we can substitute this in so that everything is in terms of P_m^2 . We can then minimize with respect to P_m^2 and then go back and find out what P_m^1 is best as well by using the relationship

$$P_m^1 = 300 - P_m^2$$

If we make this substitution we get the equation:

$$n_1 + n_2 = \frac{\ln\left(\frac{300 - P_m^2}{\left(300 - P_m^2 - \frac{31}{365}(.068)(7000)\right)}\right)}{\ln\left(1 + \frac{31}{365}(.068)\right)} + \frac{\ln\left(\frac{P_m^2}{\left(P_m^2 - \frac{31}{365}(.078)(5000)\right)}\right)}{\ln\left(1 + \frac{31}{365}(.078)\right)}$$

Now everything is in terms of one variable to minimize this lets replace P_m^2 with an x and think of it as a function then we could graph the function and find the lowest y-value for which the corresponding x-value is the optimal payment for the \$5000 loan which intern can be used to calculate the optimal payment for the \$7000 loan.

We get:

$$y = \frac{\ln\left(\frac{300 - x}{\left(300 - x - \frac{31}{365}(.068)(7000)\right)}\right)}{\ln\left(1 + \frac{31}{365}(.068)\right)} + \frac{\ln\left(\frac{x}{\left(x - \frac{31}{365}(.078)(5000)\right)}\right)}{\ln\left(1 + \frac{31}{365}(.078)\right)}$$

You are welcome to plot this using software and use it to help you find the minimum but here is also a table to help you along. Note since the minimum payment is 50 and the maximum we could make 300 we choose to restrict the domain of the function to be from x=50 to x=300.

Monthly Payment	Calculated Time(n1+n2)	(x,y) pair to plot
50		
60		
70		
80		
90		
100		
110		
120		
130		
140		
150		
160		
170		
180		
190		
200		
210		
220		
230		
240		
250		
260		
270		
280		
290		
300		

Once we have the first round done we need to narrow in on the actual best payment. Look at the numbers in the first table and determine between which two payments the actual minimum time lies. You can see this as there will be a down trend in y values followed by an upward trend. Then go in steps of 1 between those two to find the best payment. The table below will help you organize the numbers. So for example if you find that the minimum lies between $x=120$ and $x=130$ you would calculate y values for 120, 121, 122, 123, 124, 125, 126, 127, 128, 129, 130 and choose the one that gives you the smallest y value.

Monthly Payment	Calculated time

Now that you have the minimum for x recall that corresponds the payment for the \$5000 loan P_m^2 We can now find the payment we want for the \$700 dollar loan with the equation

$$300 - P_m^2 = P_m^1$$

Put the minimum values in the table below:

$P_m^1 =$	$P_m^2 =$
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Next lets actually calculate how much interest you will pay and how long until you are done paying off the loans.

Using our equations for the time to pay off the loan and for total interest paid now that we know the best monthly payments for each loan fill out the table below.

Loan 1 \$7000 at 6.8% interest	Optimal Payment:	Time until paid off:	Total Interest paid:
Loan 2 \$5000 at 7.8% interest	Optimal payment:	Time until paid off:	Total interest paid
	Total monthly payment	Time until both are paid off:	Total interest paid for both loans:

Last nut not least lets see if this helped us at all in fill out the table below assuming you made the minimum monthly payment of \$50 for both loans

Loan 1 \$7000 at 6.8% interest	Monthly payment \$50	Time until paid off:	Total Interest paid:
Loan 2 \$5000 at 7.8% interest	Monthly payment: \$50	Time until paid off:	Total interest paid
	Total monthly payment \$100	Time until both are paid off:	Total interest paid for both loans:

What I want you to do for full credit.

1. Do all of the calculation required to fill out every table in the document and fill out the tables. You can use a calculator or computer for this. (15pts)
 2. Make a nice plot using the (x,y) pairs calculated in the table on page 6. (5pts)
 3. Reproduce the work where we solved for the number of months it takes to pay off the loan justifying each step including which property of logarithms we used. (5pts)
- For an extra 5 points you can derive the total interest paid equation given using the equation for the amount owed in the nth month and the equations for geometric series in chapter 9.

Reproduce the Algebraic work here if you like.