

Transformations of functions

Let's start by considering a few functions to see how they might be related.

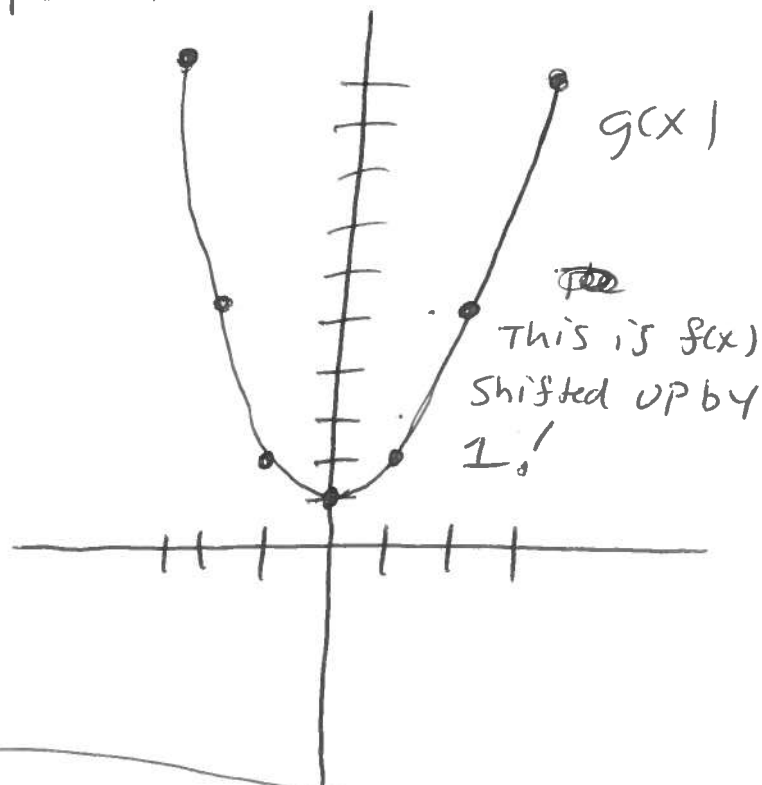
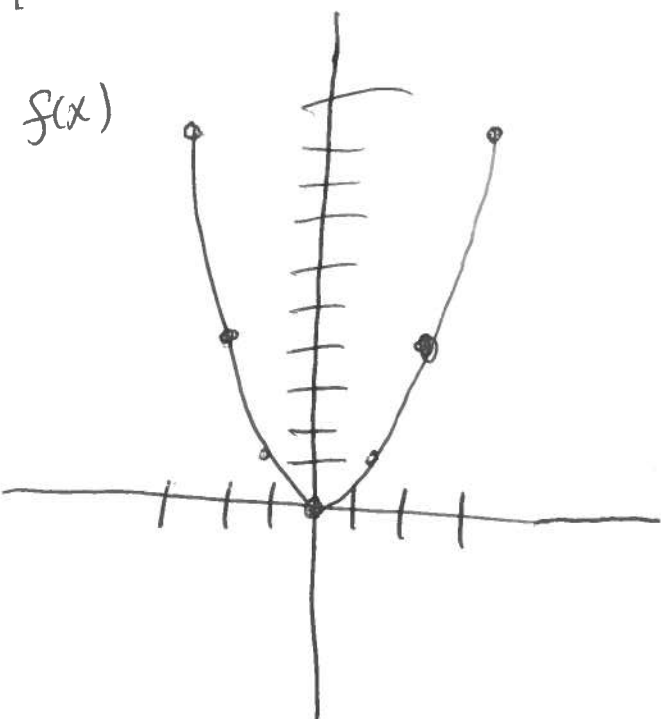
$$f(x) = x^2$$

x	$f(x) = x^2$	$(x, f(x))$
-3	$(-3)^2 = 9$	$(-3, 9)$
-2	$(-2)^2 = 4$	$(-2, 4)$
-1	$(-1)^2 = 1$	$(-1, 1)$
0	$0^2 = 0$	$(0, 0)$
1	$1^2 = 1$	$(1, 1)$
2	$2^2 = 4$	$(2, 4)$
3	$3^2 = 9$	$(3, 9)$

$$g(x) = x^2 + 1$$

x	$g(x) = x^2 + 1$	$(x, g(x))$
-3	$(-3)^2 + 1 = 10$	$(-3, 10)$
-2	$(-2)^2 + 1 = 5$	$(-2, 5)$
-1	$(-1)^2 + 1 = 2$	$(-1, 2)$
0	$0^2 + 1 = 1$	$(0, 1)$
1	$1^2 + 1 = 2$	$(1, 2)$
2	$2^2 + 1 = 5$	$(2, 5)$
3	$3^2 + 1 = 10$	$(3, 10)$

Notice all the x values are the same as $f(x) = x^2$ but the y values are all 1 bigger



Note: $g(x) = f(x) + 1 = x^2 + 1$

$$f(x) = x^2$$

$$g(x) = (x-1)^2 \quad (68)$$

x	$f(x) = x^2$	$(x, f(x))$
-3	$(-3)^2$	$(-3, 9)$
-2	$(-2)^2$	$(-2, 4)$
-1	$(-1)^2$	$(-1, 1)$
0	0^2	$(0, 0)$
1	1^2	$(1, 1)$
2	2^2	$(2, 4)$
3	3^2	$(3, 9)$

x	$g(x) = (x-1)^2$	$(x, g(x))$
-3	$(-3-1)^2 = 16$	$(-3, 16)$
-2	$(-2-1)^2 = 9$	$(-2, 9)$
-1	$(-1-1)^2 = 4$	$(-1, 4)$
0	$(0-1)^2 = 1$	$(0, 1)$
1	$(1-1)^2 = 0$	$(1, 0)$
2	$(2-1)^2 = 1$	$(2, 1)$
3	$(3-1)^2 = 4$	$(3, 4)$

Lets compare these ordered pairs and the y values

$$f(x) = x^2$$

$$g(x) = (x-1)^2$$

$$(-3, 9) \leftarrow$$

$$(-3, 16)$$

$$(-2, 4) \leftarrow$$

$$\rightarrow (-2, 9)$$

$$(-1, 1) \leftarrow$$

$$\rightarrow (-1, 4)$$

$$(0, 0) \leftarrow$$

$$\rightarrow (0, 1)$$

$$(1, 1) \leftarrow$$

$$\rightarrow (1, 0)$$

$$(2, 4) \leftarrow$$

$$\rightarrow (2, 1)$$

$$(3, 9) \leftarrow$$

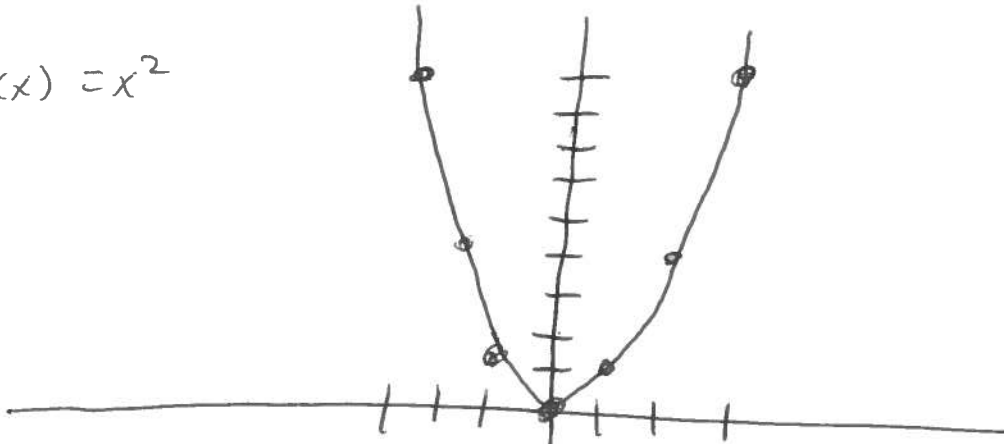
$$\rightarrow (3, 4)$$

Notice that the x values which produce the same y-values as $f(x) = x^2$ for $g(x) = (x-1)^2$ are

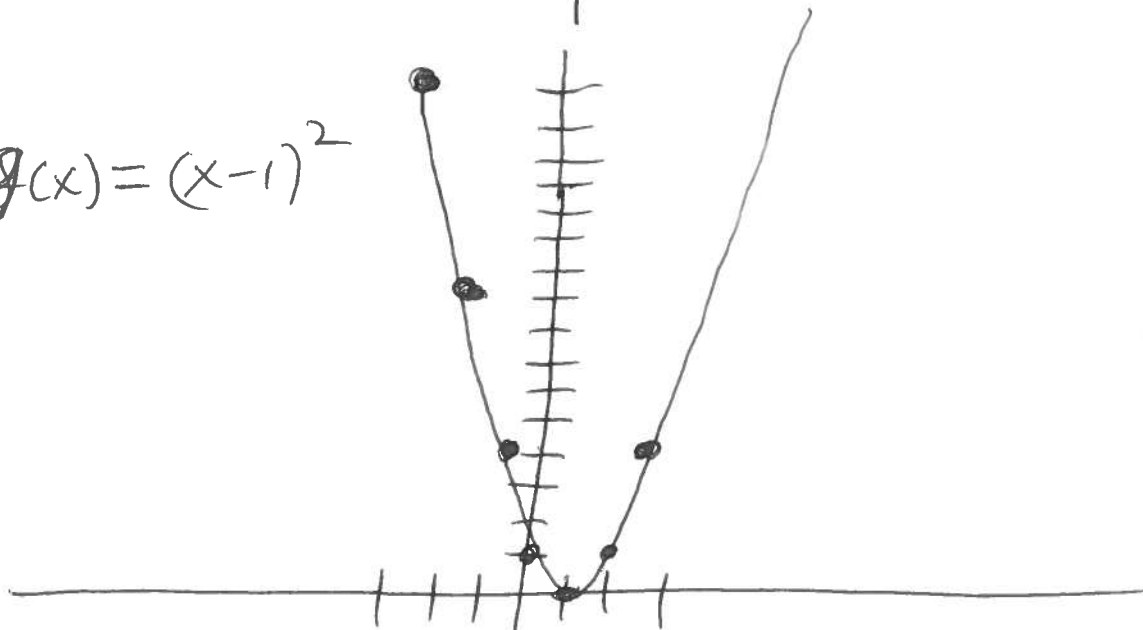
all larger by 1 [Note: $g(x) = f(x-1) = (x-1)^2$]

Plotting the functions gives us

$$f(x) = x^2$$



$$g(x) = (x-1)^2$$



This is the Graph
of $f(x) = x^2$
shifted to the right
by 1

One more.

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$$f(x) = x^2$$

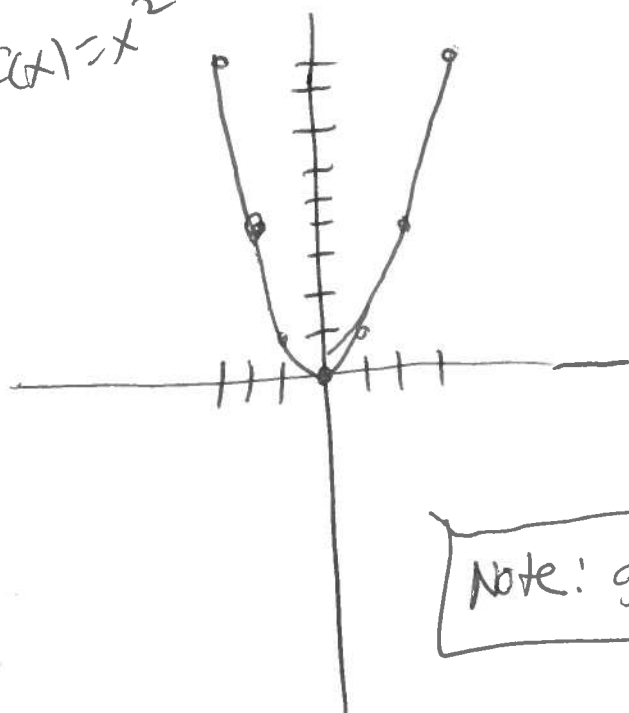
x	$f(x) = x^2$	$(x, f(x))$
-3	$(-3)^2 = 9$	$(-3, 9)$
-2	$(-2)^2 = 4$	$(-2, 4)$
-1	$(-1)^2 = 1$	$(-1, 1)$
0	$0^2 = 0$	$(0, 0)$
1	$1^2 = 1$	$(1, 1)$
2	$2^2 = 4$	$(2, 4)$
3	$3^2 = 9$	$(3, 9)$

$$g(x) = -x^2$$

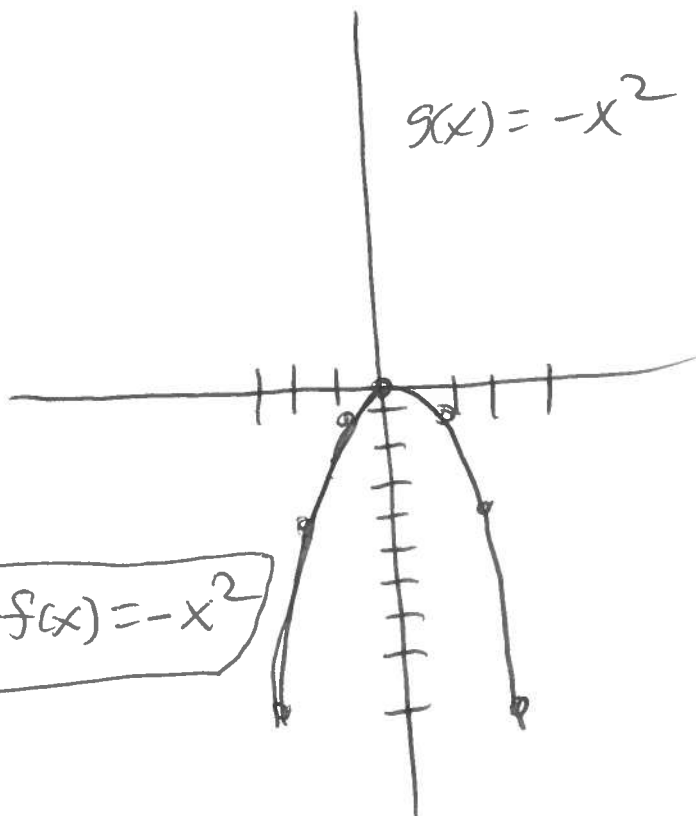
x	$g(x) = -x^2$	$(x, g(x))$
-3	$-(-3)^2 = -9$	$(-3, -9)$
-2	$-(-2)^2 = -4$	$(-2, -4)$
-1	$-(-1)^2 = -1$	$(-1, -1)$
0	$-0^2 = 0$	$(0, 0)$
1	$-(1)^2 = -1$	$(1, -1)$
2	$-(2)^2 = -4$	$(2, -4)$
3	$-(3)^2 = -9$	$(3, -9)$

Notice that the y -values for $g(x)$ are just the negatives of those for $f(x) = x^2$

$$f(x) = x^2$$



$$g(x) = -x^2$$



Note: $g(x) = -f(x) = -x^2$

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Some types of transformations

Let c be a positive real number for a parent function $f(x)$ the graph of $f(x)$ moves in the following ways.

Vertical shift c units up: $f(x) + c$

Vertical shift c units down: $f(x) - c$

Horizontal shift right c units: $f(x - c)$

Horizontal shift left c units: $f(x + c)$

Reflection over x -axis: $-f(x)$

Reflection over y -axis: $f(-x)$

Anatomy of a transformation

consider the parent function $f(x) = |x|$
what transformations have taken place with

$$g(x) = \underbrace{-}_{\substack{\uparrow \\ \text{reflection} \\ \text{over } x\text{-axis}}} |x| \underbrace{- 3}_{\substack{\uparrow \\ \text{shift} \\ \text{right} \\ 3}} \underbrace{+ 4}_{\substack{\uparrow \\ \text{shift} \\ \text{up} \\ 4}}$$

To see this let's go through the transformations 72

$$f(x) = |x|$$

reflect over x-axis $-f(x)$

$$-f(x) = -|x|$$

Now

Shift right 3 replace x with $x-3$

$$-f(x-3) = -|x-3|$$

Now shift up 4 by adding 4

$$-f(x-3) + 4 = -|x-3| + 4 = g(x)$$

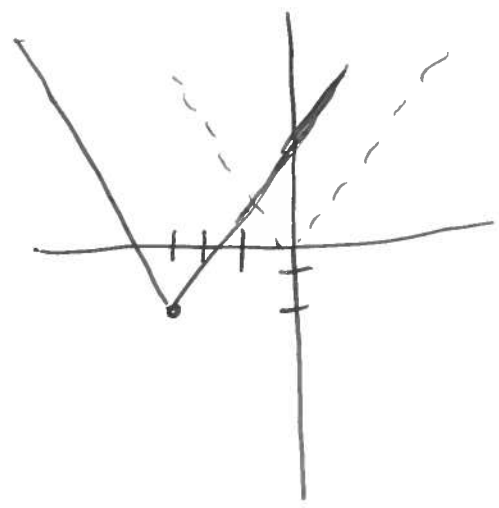
One of the best ways to get a handle on this is to see lots of examples...

For the parent function $f(x) = |x|$ Graph

$$|x+3|-2$$

shift left 3

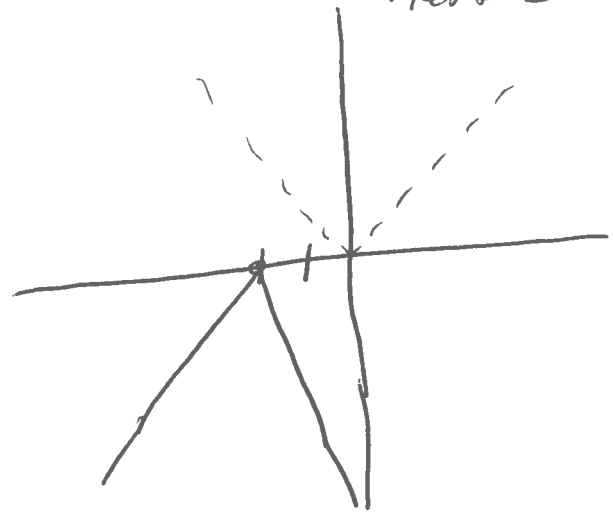
shift down 2



reflection over x

$$-|x+2|$$

left 2

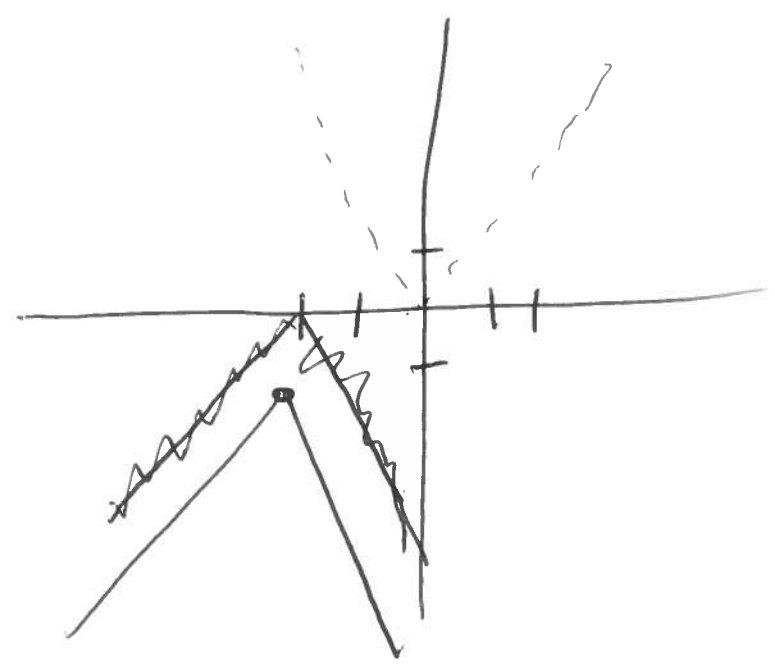


reflect
shifted
graph

shift left 2

$$-(|x+2|+1)$$

up 1

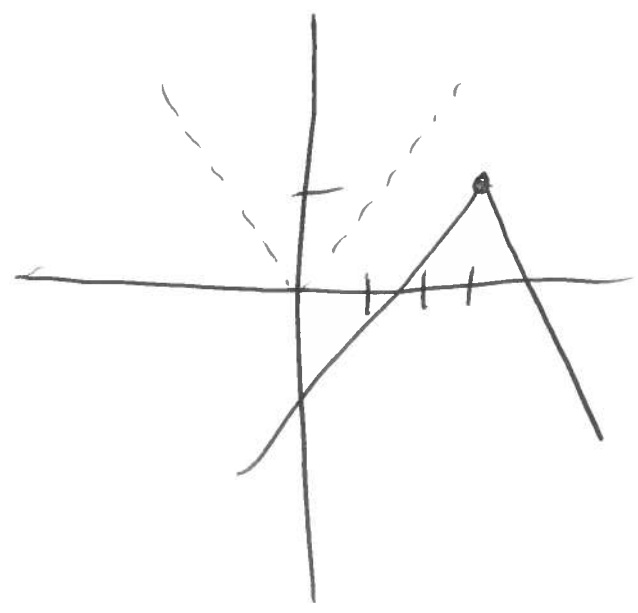


$$-|-3|+1$$

reflect
over x

right
3

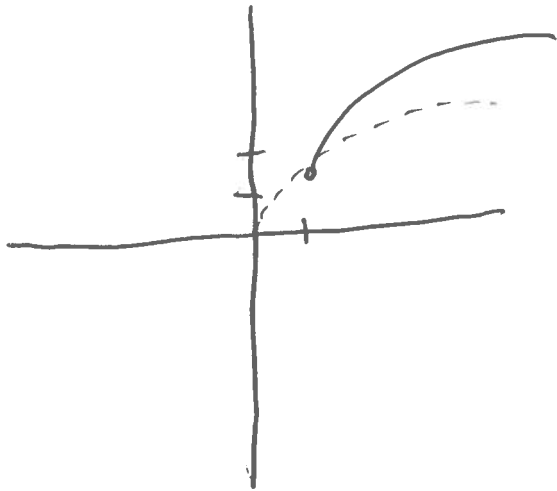
up 1



for the parent function $f(x) = \sqrt{x}$ graph the following

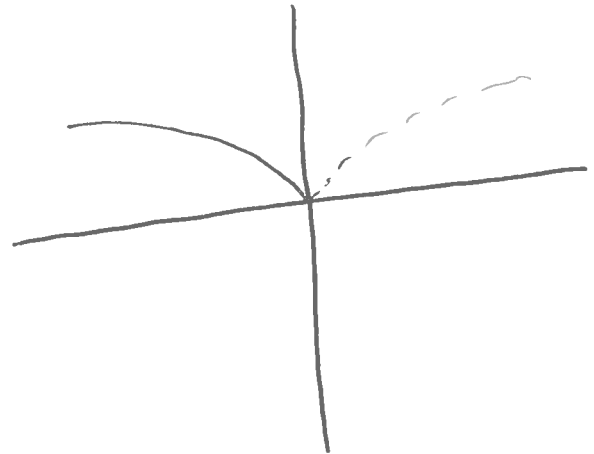
$$\sqrt{x-1} + 2$$

↑ right 1
↑ UP 2



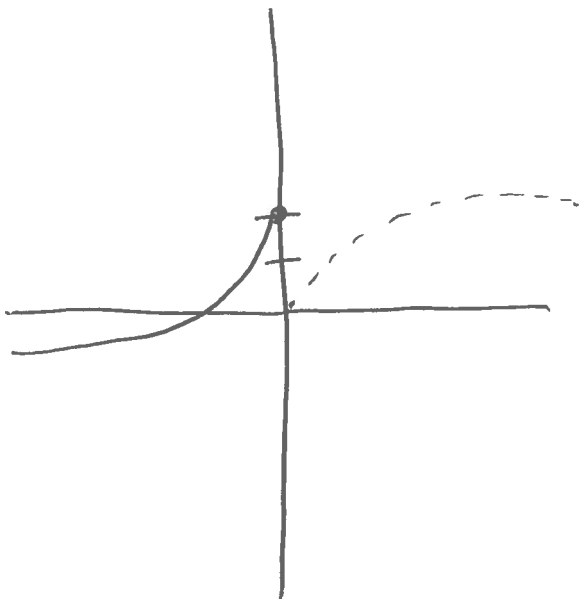
$$\sqrt{-x}$$

↑ reflect over y axis



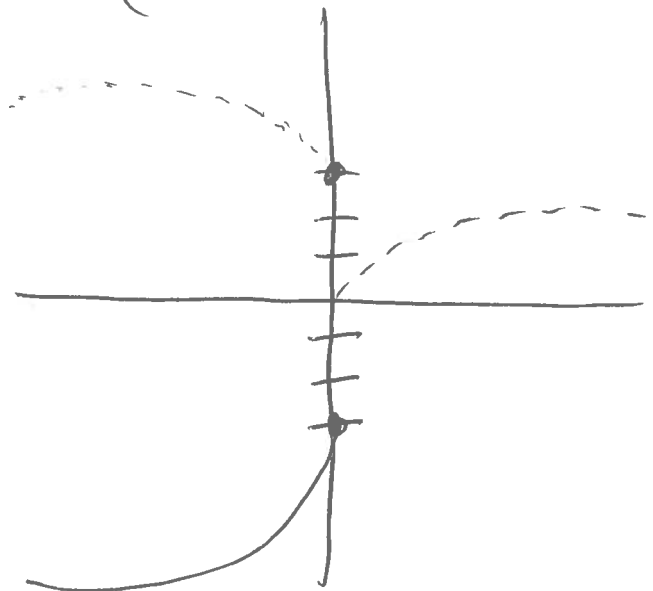
$$-\sqrt{x} + 2$$

reflect over x
↓
UP 2



$$-(\sqrt{x} + 3)$$

reflect shifted graph over x
↓
UP 3



Nonrigid transformations

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for the parent function $f(x)$ and $c > 1$

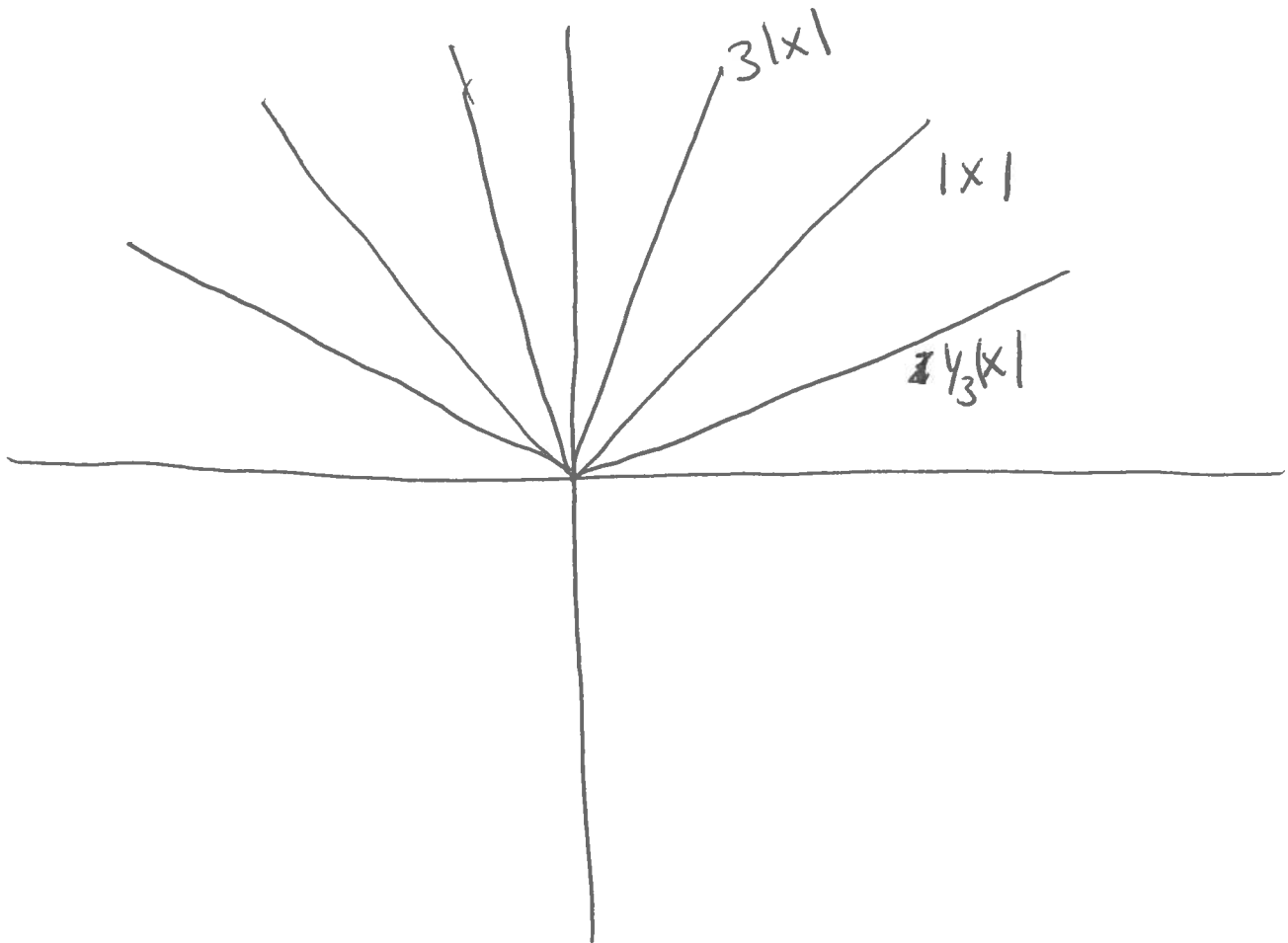
$cf(x)$ is a vertical stretch

$f(cx)$ is a horizontal shrink

when $0 < c < 1$

$cf(x)$ is a vertical shrink

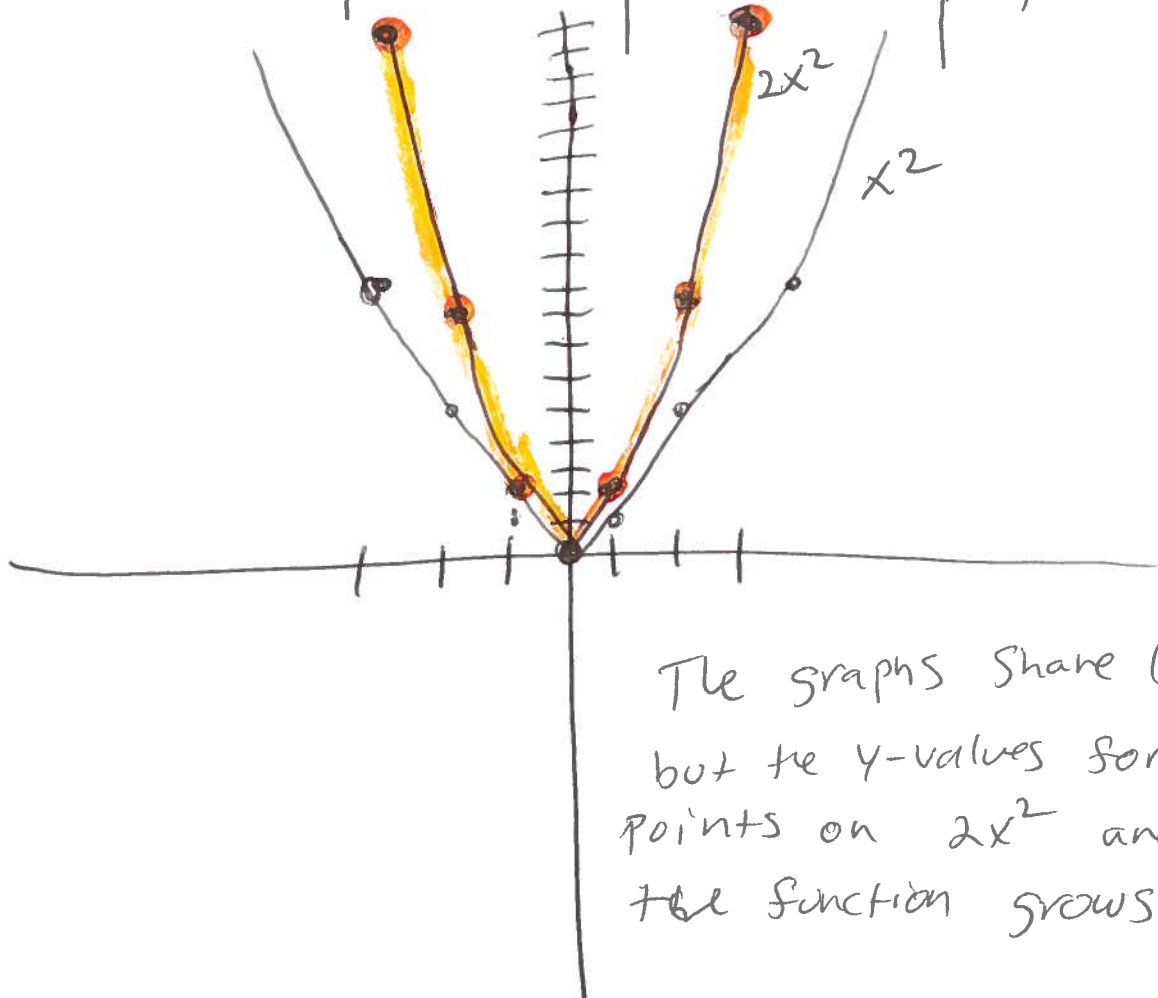
$f(cx)$ is a horizontal stretch



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To see this consider $f(x) = x^2$ again
and $g(x) = 2x^2$

	$f(x) = x^2$	$(x, f(x))$	$g(x) = 2x^2$	$(x, g(x))$
-3	$(-3)^2 = 9$	-3, 9	$2(-3)^2 = 18$	-3, 18
-2	$(-2)^2 = 4$	-2, 4	$2(-2)^2 = 8$	-2, 8
-1	$(-1)^2 = 1$	-1, 1	$2(-1)^2 = 2$	-1, 2
0	$0^2 = 0$	0, 0	$2(0)^2 = 0$	0, 0
1	$1^2 = 1$	1, 1	$2(1)^2 = 2$	1, 2
2	$2^2 = 4$	2, 4	$2(2)^2 = 8$	2, 8
3	$3^2 = 9$	3, 9	$2(3)^2 = 18$	3, 18



The graphs share $(0,0)$
but the y-values for all other
points on $2x^2$ are larger
the function grows faster,