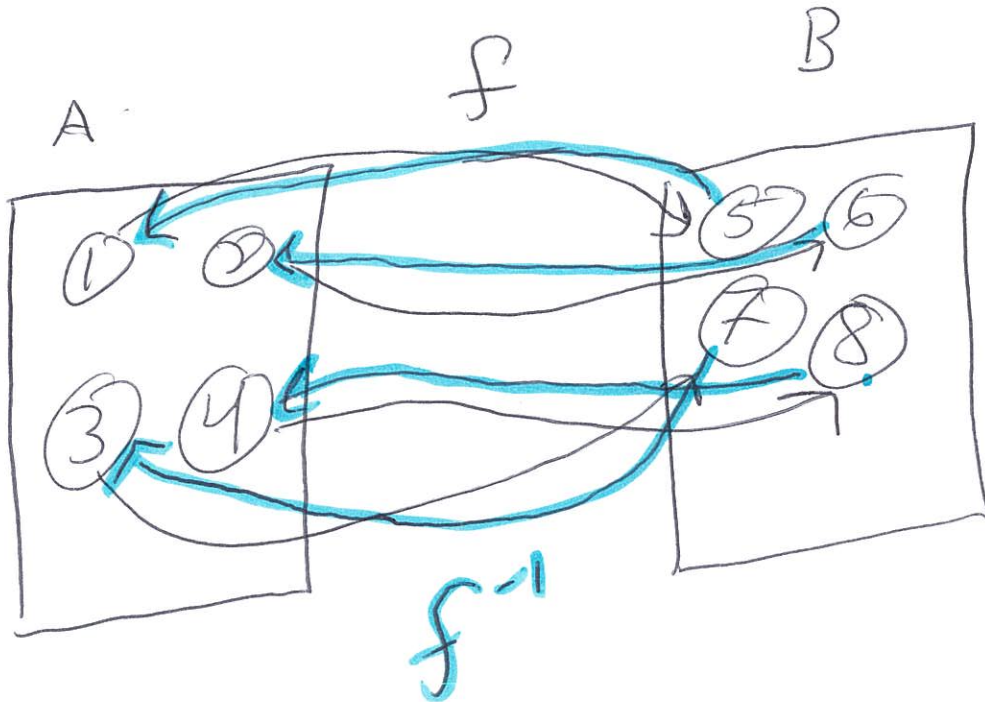
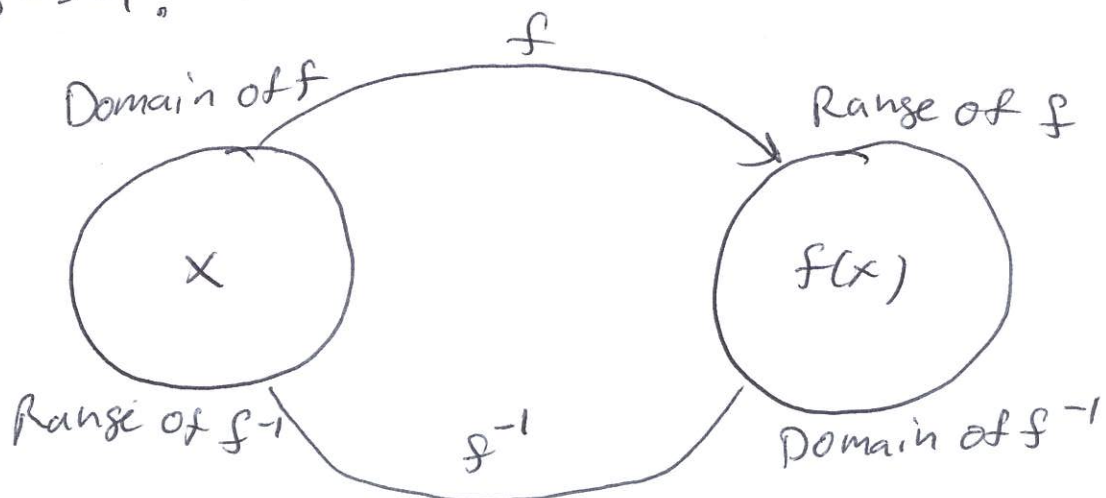


Inverse Functions



Suppose we have the function $f: A \rightarrow B$ above
 f takes $1 \rightarrow 5$ $2 \rightarrow 6$ $3 \rightarrow 7$ $4 \rightarrow 8$ we define
 f^{-1} to be the function that takes us back
 that is ~~$f^{-1}: A \rightarrow B$~~ $f^{-1}: B \rightarrow A$ and maps
 us back so f^{-1} takes $5 \rightarrow 1$ $6 \rightarrow 2$ $7 \rightarrow 3$
 $8 \rightarrow 4$.



Doing and undoing.

Suppose we have $f(x) = 2x + 1$ for a given x we get a y -value by multiplying x by 2 and adding 1. What if we take a y -value of 5 and want to know its x -value well we can get there by undoing what we did to x to get to y .

Ex

$$\text{Let } x = 2 \text{ then } f(x) = y = 2(2) + 1 = 5$$

Let's undo f

$$5 = 2(2) + 1$$

$$5 - 1 = 2(2) + \cancel{1} - \cancel{1}$$

$$\frac{4}{2} = \frac{2(2)}{2}$$

$$2 = 2 = x$$

So to get back to $x=2$ from $y=5$ we subtract 1 and divide by 2 the opposite of what we did ^{in reverse} to get $y=5$ from $x=2$ namely multiply by 2 then add 1

(84)

So we could write down $f^{-1}(x)$ as take the
Given number x subtract 1 and divide by 2

$$f^{-1}(x) = \frac{x-1}{2}$$

So for

$$f(x) = 2x+1$$

$$f^{-1}(x) = \frac{x-1}{2}$$

Ex

Find the inverse for $f(x) = \frac{1}{4}x - 2$

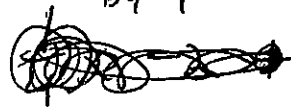
$$f(x) = \frac{1}{4}x - 2$$

multiply

by $\frac{1}{4}$ so
multiply
by 4

we subtract 2

so to undo this add 2



$$f^{-1}(x) = 4(x+2)$$

Ex

What is the inverse of $f(x) = x^2$ $x \geq 0$

Here we are squaring we can undo
this by taking the square root

$$f^{-1}(x) = \sqrt{x}$$

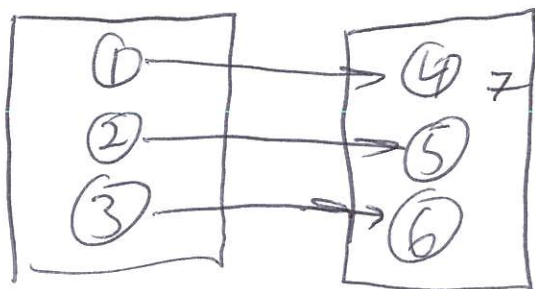
~~when~~

when we don't have an inverse function

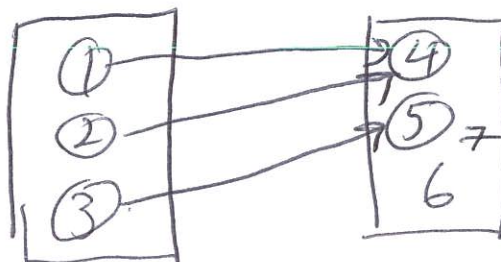
In order to have an inverse function for a given function we need f to be what is called one to one. A function f is one to one when each value in the domain is matched with a unique value in the range.

Ex

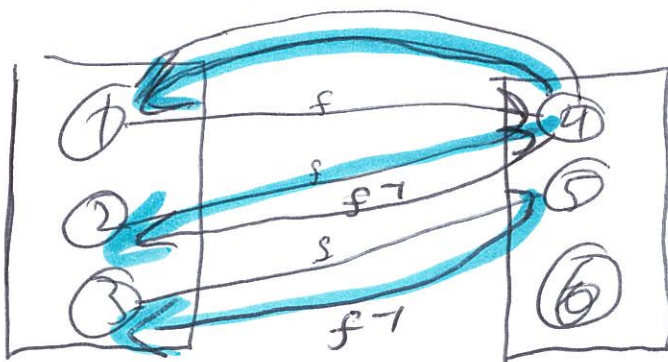
one to one



NOT one to one



Why can't we have an inverse if f is not one to one

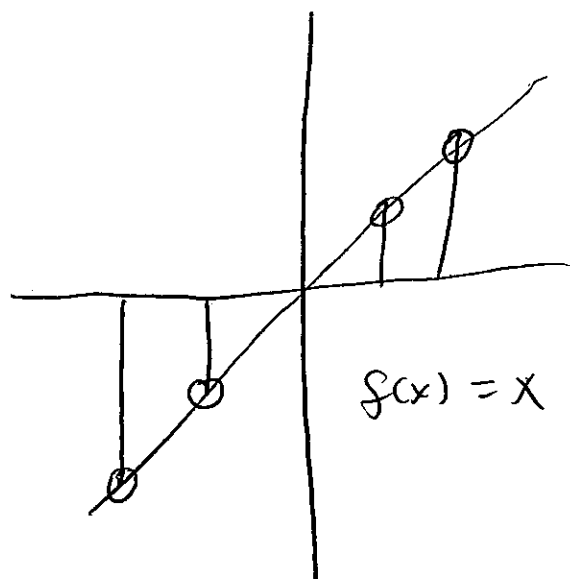


In this example f^{-1} would map 4 to two values which means it is NOT a function,

Examples

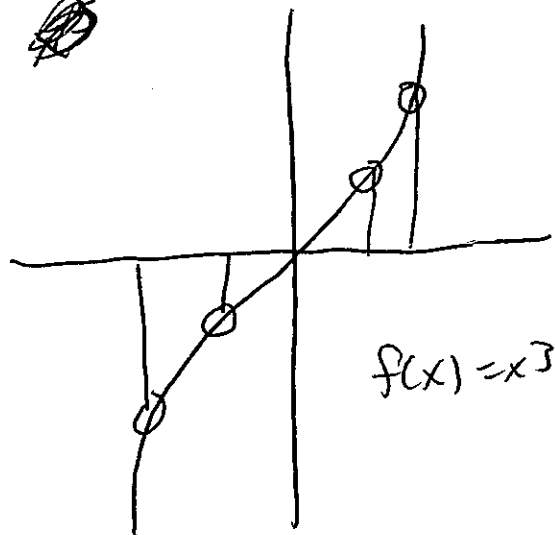
86

one to one



$$f(x) = x$$

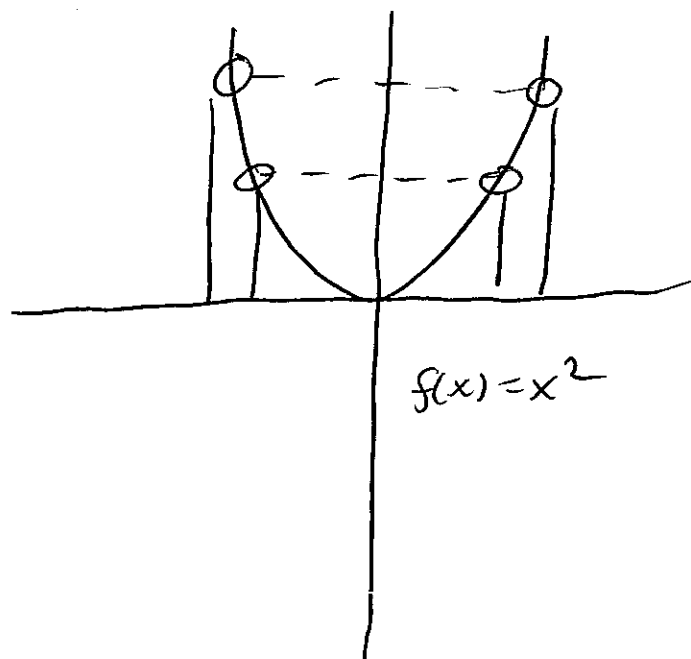
~~87~~



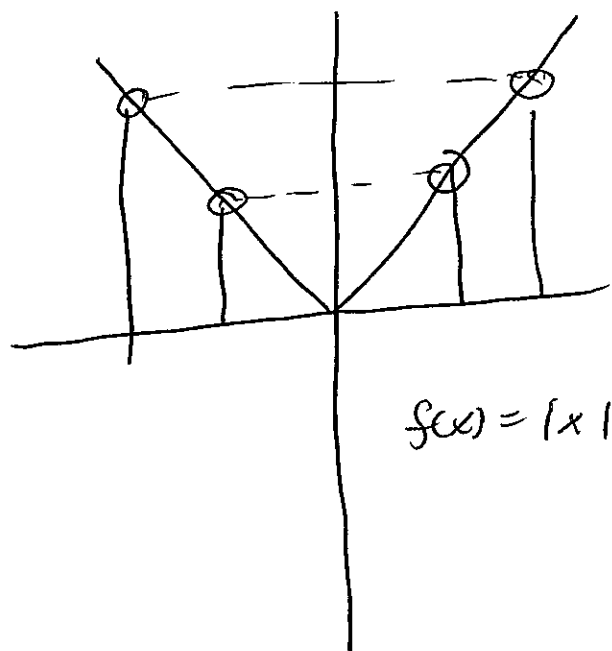
$$f(x) = x^3$$

Note ~~88~~ 1-1 functions
pass the horizontal
line test

Not one to one



$$f(x) = x^2$$

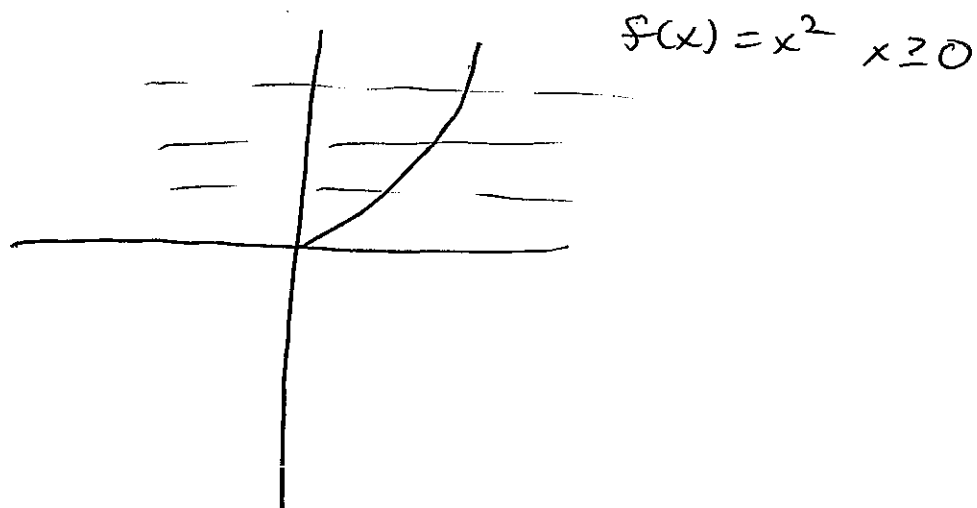


$$f(x) = |x|$$

Functions that are
not 1-1 fail the horizontal
line test.

Restricting the domain to have an inverse.

$f(x) = x^2$ is not 1-1 so we don't have an inverse function unless we only look at part of the domain say $x \geq 0$



When we include only the right half of the graph $f(x) = x^2$ is one-to-one and passes the horizontal line test. In

this case we have an inverse

which is given by $f^{-1}(x) = \sqrt{x}$

HW Example #58 HW 1

Which of the following functions are one-one

$$f(x) = \frac{6}{x}$$

$$f(x) = |x - 9|$$

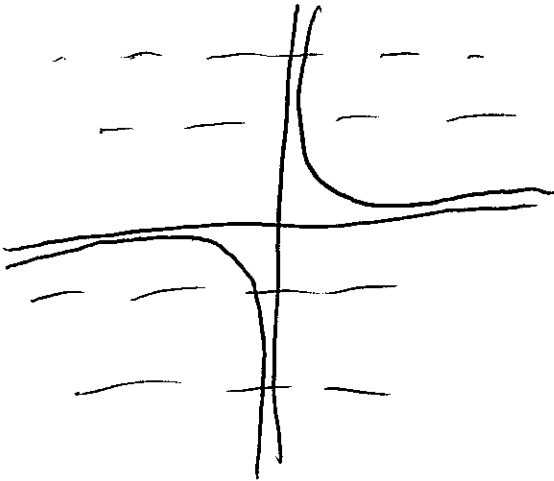
$$f(x) = x^4 + 5 \quad 0 \leq x \leq 2$$

$$f(x) = x^3 - x \quad \text{on their domains}$$

Soln

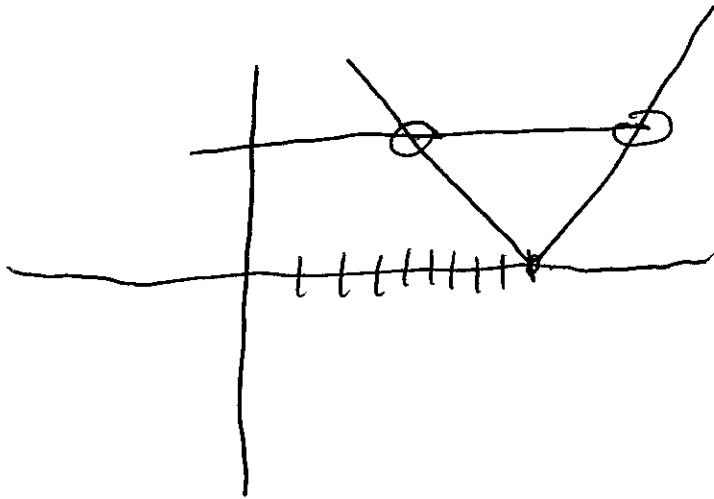
$$f(x) = 6/x$$

Look at the graph



passes the horizontal
line test so this
function is 1-1

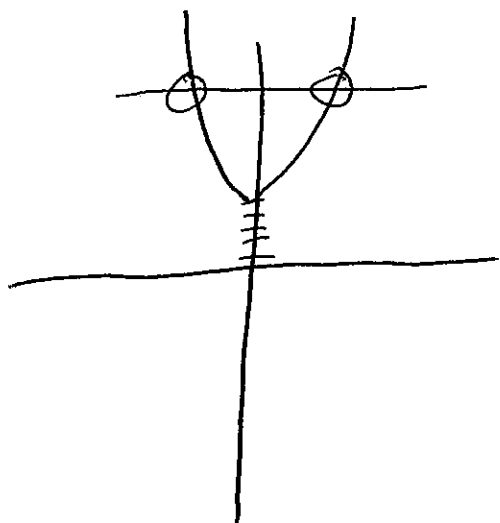
$$f(x) = |x-9|$$



Fails horizontal
line test

not 1-1

$$f(x) = x^4 + 5$$



Fails H-L test not

1-1 Also choose $x = -1$
 $x = 1$

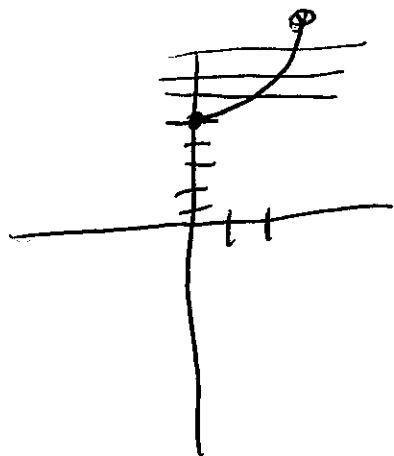
$$f(1) = 1^4 + 5 = 6$$

$$f(-1) = (-1)^4 + 5 = 6$$

Two x -values go to the same y -value

However if we restrict our domain

$$f(x) = x^4 + 5 \quad 0 \leq x \leq 2$$



Now we pass the
H-L line test so we
can say over $0 \leq x \leq 2$
the function is 1-1

$$f(x) = x^3 - x$$

This one we might not be sure how to graph right away so let's try something else

Suppose x_1, x_2 are in the domain of $f(x)$ and $x_1 \neq x_2$ let's see if

$f(x_1) = f(x_2)$ is possible

$$f(x_1) = x_1^3 - x_1 \quad f(x_2) = x_2^3 - x_2$$

we will show $f(x_1) \neq f(x_2)$ by assuming it does and finding that because of that assumption we find something else that is false. This is called proof by contradiction.

Suppose $x_1 \neq x_2$ but $f(x_1) = f(x_2)$

Then
$$x_1^3 - x_1 = x_2^3 - x_2$$

$$\Rightarrow x_1(x_1^2 - 1) = x_2(x_2^2 - 1)$$

$$\Rightarrow \frac{x_1(x_1^2-1)}{(x_1^2-1)} = \frac{x_2(x_2^2-1)}{(x_2^2-1)} \quad \text{divide by } x_2^2-1 \quad (91)$$

$$\Rightarrow \frac{x_1}{x_1} \frac{(x_1^2-1)}{(x_2^2-1)} = \frac{x_2}{x_1} \quad \text{divide by } x_1$$

$$\Rightarrow \frac{x_1^2-1}{x_2^2-1} = \frac{x_2}{x_1}$$

Now since $x_1 \neq x_2$ it is impossible for $\frac{x_2}{x_1} = 1$ Therefore

$$\frac{x_1^2-1}{x_2^2-1} = \frac{x_2}{x_1} \neq 1$$

$$\Rightarrow \frac{x_1^2-1}{x_2^2-1} \neq 1$$

Now the kicker

Choose $x_1 = 2$

$$x_2 = -2$$

Then $x_1 \neq x_2$ but ----

$$\begin{aligned} & \begin{matrix} x_1 \\ \swarrow \end{matrix} \frac{2^2-1}{(-2)^2-1} = \frac{4-1}{4-1} = \frac{3}{3} = 1 \end{aligned}$$

what we have just shown is (72)
that assuming $f(x_1) = f(x_2)$ ~~and~~ ^{and} $x_1 \neq x_2$ ment

That $\frac{x_1^2 - 1}{x_2^2 - 1} \neq 1$ but we can choose

$$x_1 = 2 \quad x_2 = -2 \Rightarrow \frac{x_1^2 - 1}{x_2^2 - 1} = 1$$

This is a contradiction thus the
initial assumption that $f(x_1) = f(x_2)$
when $x_1 \neq x_2$ is false Thus when

$x_1 \neq x_2$ $f(x_1) \neq f(x_2)$ which means
the function is 1-1,

— phew!

You could also graph it using
a calculator or table of values
to figure this out-----

Finding an inverse function Algebraically

(90)

HW example

$$f(x) = 3x + 11 \quad \# 63$$

$$\sqrt{x-5} \quad \# 65$$

$$ax+b \quad \# 66$$

$$\frac{x_1}{x_2} = \frac{x_1^2 - 1}{x_2^2 - 1}$$

$$x_1^3 - x_1 = x_2^3 - x_2$$

$$1 \neq \frac{x_1^2 - 1}{x_2^2 - 1}$$

$$\cancel{0} \quad x_1(x_1^2 - 1) = x_2(x_2^2 - 1) \quad \text{choose}$$

$$x_1 = 2$$

$$x_2 = -2$$

$$\Rightarrow \frac{x_1}{x_2} = \frac{x_2^2 - 1}{x_1^2 - 1}$$

$$\frac{x_1^2 - 1}{x_2^2 - 1} = 1$$

$$\frac{x_1}{x_2} \neq 1 \text{ since } x_1 \neq x_2$$

Contradiction

$$\Rightarrow \frac{x_2^2 - 1}{x_1^2 - 1} \neq 1 \quad x_2^2 \neq x_1^2$$

$$\Rightarrow \frac{x_2^2 - 1}{x_1^2 - 1} \neq 1 \Rightarrow x_1 \neq x_2$$

Finding an inverse function Algebraically,

93

Given a function $f(x)$ we can find its
inverse with the following steps. EX $f(x) = 3x + 4$

① Replace $f(x)$ with y $y = 3x + 4$

② swap x and y $x = 3y + 4$

③ solve for y

$$\begin{array}{r} x = 3y + 4 \\ -4 \quad -4 \end{array}$$

$$\frac{x-4}{3} = \frac{3y}{3}$$

$$\frac{x-4}{3} = y$$

4 Replace y with $f^{-1}(x)$

$$f^{-1}(x) = \frac{x-4}{3}$$

(94)

HW Example like #65

Find the Inverse function for

$$f(x) = \sqrt{x-5} \quad x \geq 5$$

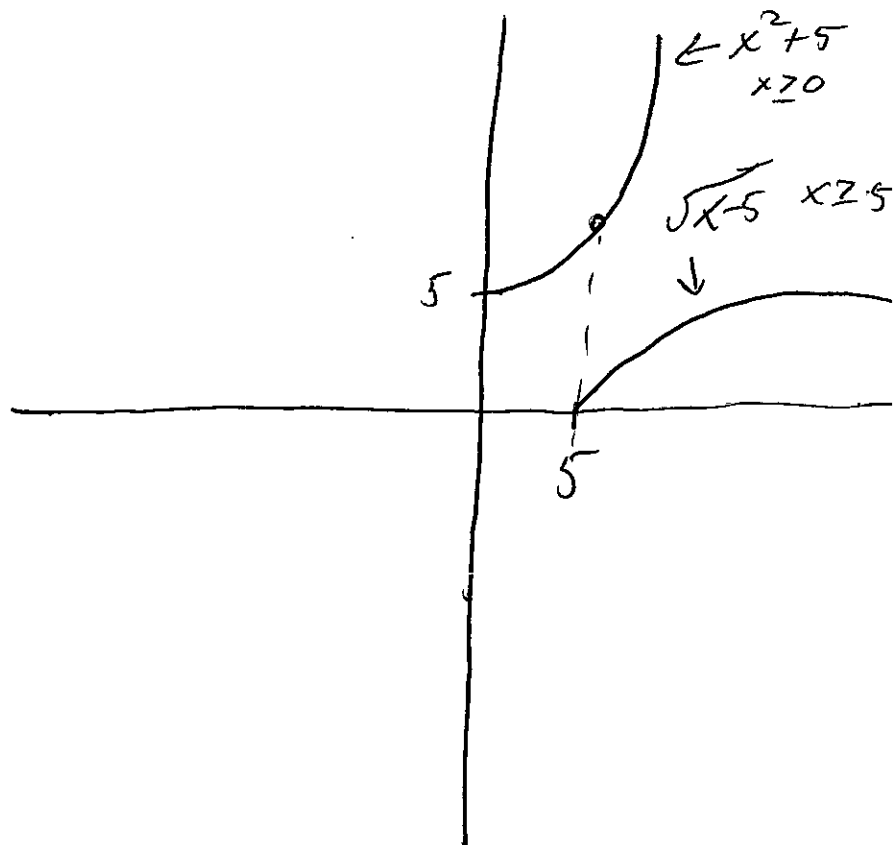
$$\textcircled{1} \quad y = \sqrt{x-5}$$

$$\textcircled{2} \quad x = \sqrt{y-5}$$

$$\textcircled{3} \quad \begin{array}{cc} x^2 = y - 5 \\ +5 \quad +5 \end{array}$$

$$x^2 + 5 = y$$

$$\textcircled{4} \quad \boxed{f^{-1}(x) = x^2 + 5}$$



HW Example like #66

Find the Inverse for ~~the~~ $f(x) = ax + b$

$$\textcircled{1} \quad y = ax + b$$

$$\textcircled{2} \quad x = ay + b$$

$$\textcircled{3} \quad \begin{array}{cc} x = ay + b \\ -b \quad -b \end{array}$$

$$x - b = ay$$

$$\frac{x-b}{a} = \frac{ay}{a}$$

$$y = \frac{x-b}{a}$$

$$\textcircled{4} \quad \boxed{f^{-1}(x) = \frac{x-b}{a}}$$

Ex Find the inverse of

95

$$f(x) = \frac{5-x}{3x+2}$$

① $y = \frac{5-x}{3x+2}$

② $x = \frac{5-y}{3y+2}$

③ $(3y+2)x = \frac{5-y}{(3y+2)} (3y+2)$

← multiply both sides by $3y+2$

$x(3y+2) = 5-y$

← distribute the x

$$\begin{array}{rcl} 3xy + 2x & = & 5 - y \\ +y & & +y \end{array}$$

$$\begin{array}{rcl} 3xy + 2x + y & = & 5 \\ -2x & & -2x \end{array}$$

$$3xy + y = 5 - 2x$$

Here we are getting all terms with a y one one side and those with out on the other

$$y(3x+1) = 5-2x$$

← factor out the y

$$y = \frac{5-2x}{3x+1} = f^{-1}(x)$$

← divide by $3x+1$ and we have it!

Graphs of functions and their inverses.

When we create an inverse function we are effectively swapping the x - y pairs that make up the graph of a function.

For example if f takes $2 \rightarrow 3$ then

the point $(2, 3)$ is on the Graph

f^{-1} will then take $3 \rightarrow 2$ which means

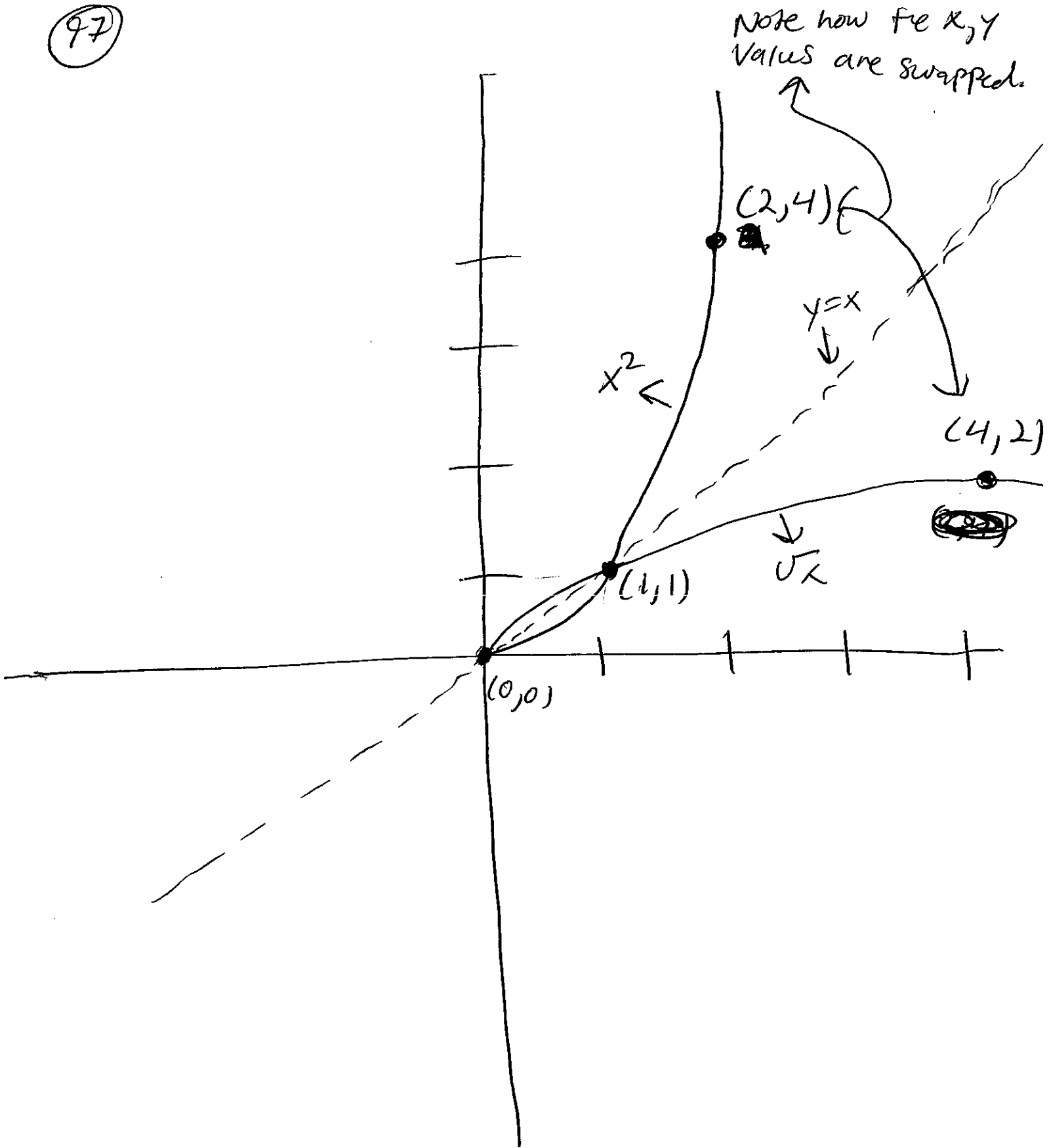
$(3, 2)$ is on the graph. So we are effectively just swapping the places of the x and y coordinates, this has the same effect as ~~swapping~~ reflecting the original function over the line $y=x$.

EX

Plot $f(x) = x^2$ and $f^{-1}(x) = \sqrt{x}$

for $x \geq 0$ on the same graph.

97



Example like HW #1 problem #

98

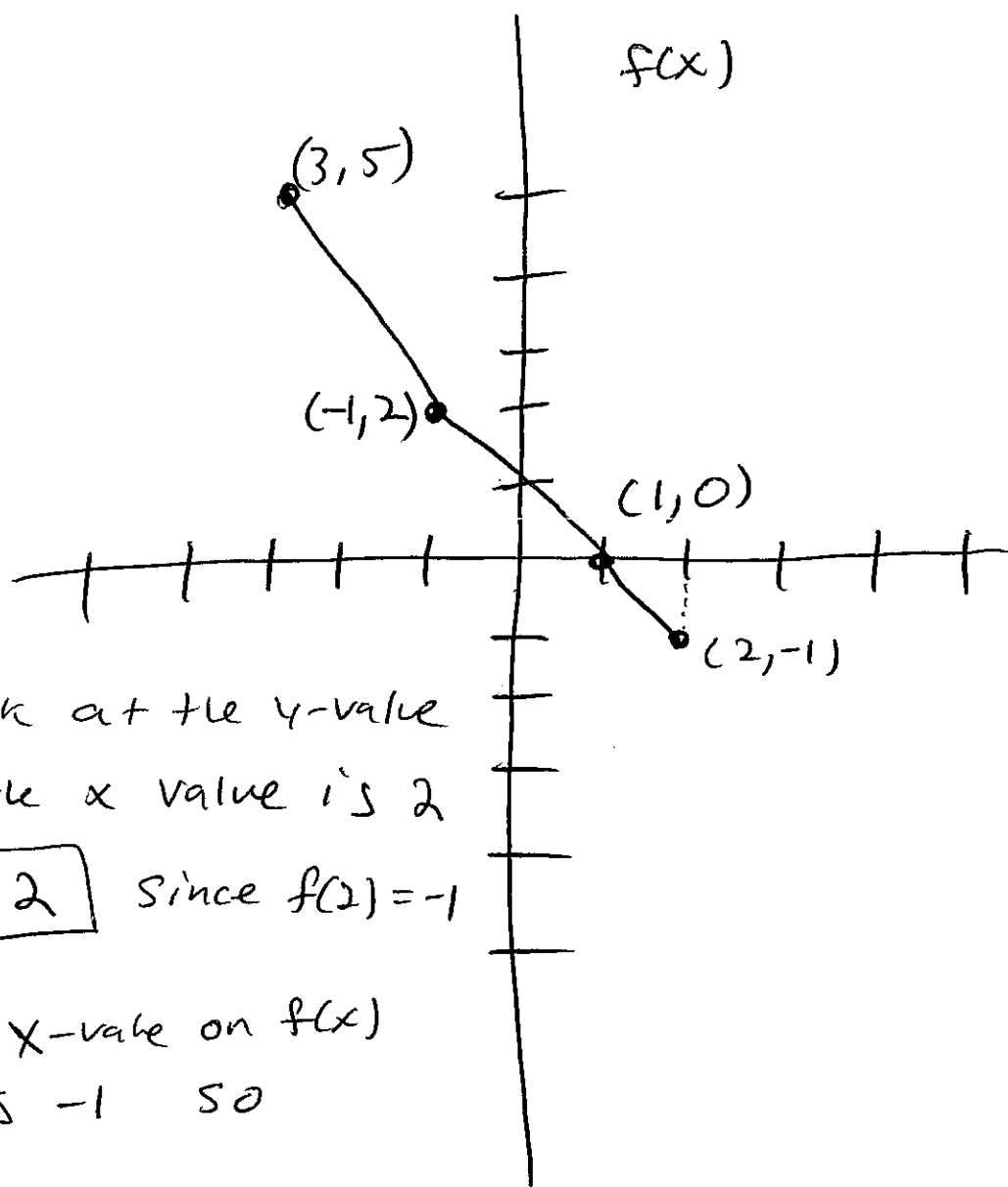
Use the graph of $f(x)$ Given to find

$$f^{-1}(-1)$$

$$f^{-1}(2)$$

$$f^{-1}(0)$$

$$f^{-1}(5)$$



For $f^{-1}(-1)$ look at the y-value
-1 on $f(x)$ the x value is 2
so $\boxed{f^{-1}(-1) = 2}$ since $f(2) = -1$

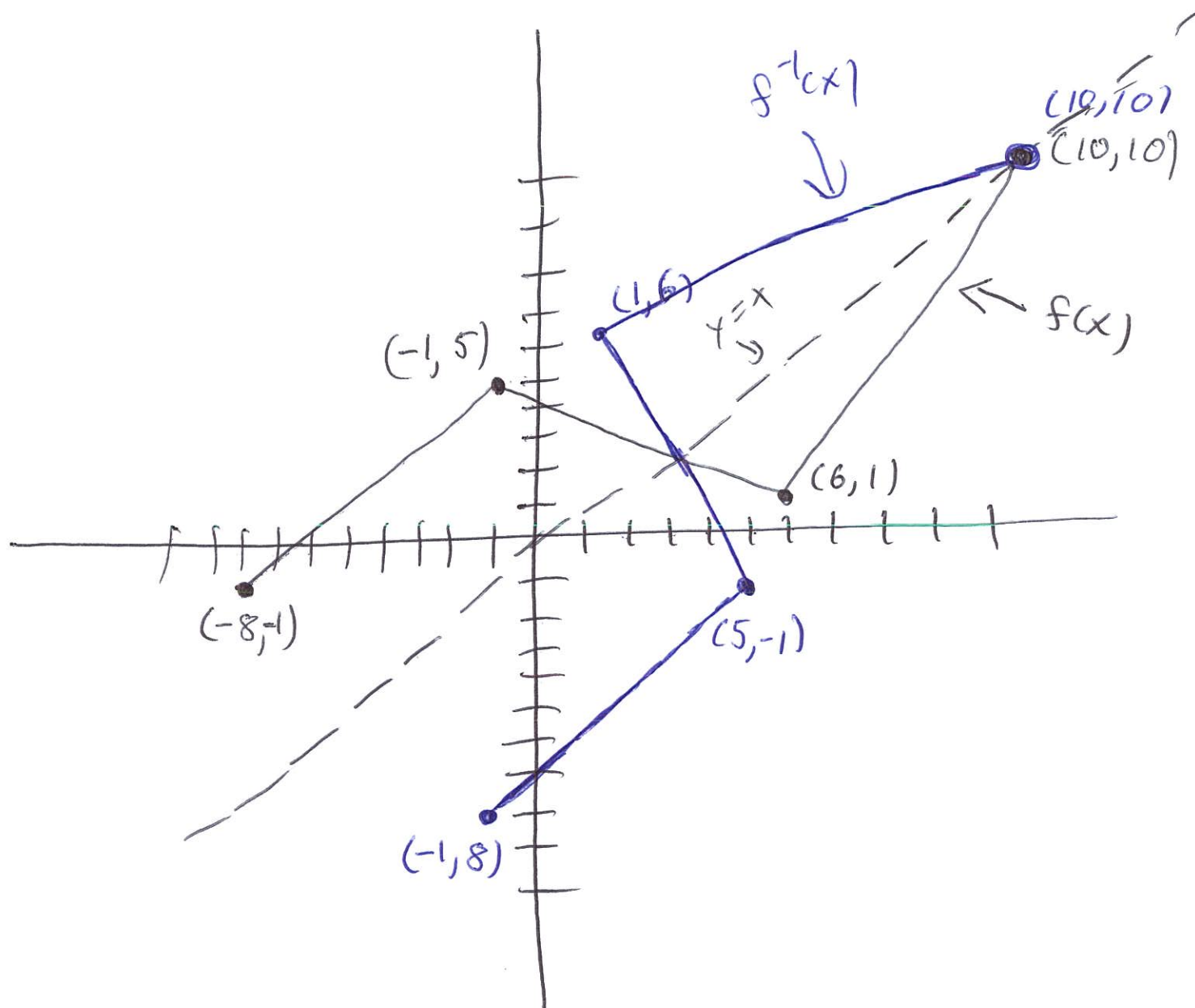
for $f^{-1}(2)$ the x-value on $f(x)$
when $y = 2$ is -1 so
 $\boxed{f^{-1}(2) = -1}$

for $f^{-1}(0)$ the x value on $f(x)$
when $y = 0$ is 1 so $\boxed{f^{-1}(0) = 1}$

for $f^{-1}(5)$ the x value on $f(x)$
when $y = 5$ is 3 so $\boxed{f^{-1}(5) = 3}$

Use the Graph of $f(x)$ to draw $f^{-1}(x)$

(99)



If

$(-8, -1)$ is on f then $(-1, -8)$ is on f^{-1}

$(-1, 5)$ is on f then $(5, -1)$ is on f^{-1}

$(6, 1)$ is on f then $(1, 6)$ is on f^{-1}

$(10, 10)$ is on f then $(10, 10)$ is on f^{-1}

Important Property of inverse functions.

100

If $f(x)$ and $f^{-1}(x)$ are inverses then

$$f \circ f^{-1}(x) = f(f^{-1}(x)) = x$$

$$f^{-1} \circ f(x) = f^{-1}(f(x)) = x$$

Ex

$$f(x) = 3x + 4 \quad f^{-1}(x) = \frac{x-4}{3}$$

$$\begin{aligned} f(f^{-1}(x)) &= 3\left(\frac{x-4}{3}\right) + 4 = \cancel{3\left(\frac{x-4}{3}\right)} + 4 \\ &= x - 4 + 4 = x \quad \checkmark \end{aligned}$$

$$\begin{aligned} f^{-1}(f(x)) &= \frac{f(x) - 4}{3} = \frac{3x + \cancel{4} - 4}{3} \\ &= \frac{3x}{3} = x \quad \checkmark \end{aligned}$$

When ever you compose a function with its inverse you get only the independent variable back.