

Linear factorization theorem

If $f(x)$ ~~is~~ is a polynomial of degree n where $n > 0$, then $f(x)$ has precisely n linear factors

$$f(x) = a_n(x-c_1)(x-c_2)(x-c_3)\dots(x-c_n)$$

HW Example. Like # 28 HW 2

$$\text{Let } f(x) = x^3 - 8$$

Is $(x-8)$ a factor of $f(x)$?

Is $(x-2)$ a factor of $f(x)$?

is $(x+2)$ a factor of $f(x)$?

Soln

$$0 = x^3 - 8 \Rightarrow x^3 = 8 \Rightarrow x = 8^{1/3} = 2$$

Since 2 is a zero of $x^3 - 8$

$x-2$ is a factor.

HW Example like # 42 HW 2

Find a degree 3 polynomial with leading coeff 1 and zeros $-4, -5i, 5i$

Soln

$$(x - (-4))(x - (-5i))(x - 5i) = (x+4)(x+5i)(x-5i)$$

The Rational Zero test

If the polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$ has integer coefficients then every rational

Zero of f has the form $\frac{p}{q}$ where

p, q have no common factors other

than 1 and p is a factor of a_0 ^{constant term}

q is a factor of a_n ^{leading term}

How Example like #40 How 2

Let $f(x) = x^4 - x^3 - 12x^2 - 3x + 1 = 0$ ~~be~~ Find the list of all potential rational solutions then the actual solutions.

Soln

We apply the rational root theorem,

$a_n = 1$ $a_0 = 1$ here the factors of

a_n are only ± 1 and a_0 only ± 1 so the

possible roots are 1 or -1 to check

we plug them into the Equation

$$1^4 - 1^3 - 12(1) - 3(1) + 1 =$$

$$(-1)^4 - (-1)^3 - 12(-1) - 3(-1) + 1 =$$

So we have no rational roots we instead have either irrational or complex roots

EX like HW #2 ~~145~~ #45

Find all the real zeros of the polynomial

$$P(x) = x^4 - 13x^2 - 12x$$

Soln

First we can factor out an x

$$x(x^3 - 13x - 12) = 0$$

Thus $x=0$ is a solution next we will use the rational root theorem to find the zeros of $x^3 - 13x - 12$

$a_n = 1$ $a_0 = -12$ we will list

all of the factors of a_n a_0 to build a list of possible roots.

(4)

$$q_n = 1 \Rightarrow q = 1 \text{ or } -1$$

$$a_n = 12 \Rightarrow p = \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$$

So our possible roots are $\frac{p}{q}$

Lets ~~check each of these~~ through the $\frac{p}{q}$'s

$$\frac{p}{q} = \pm \frac{1}{1}, \pm \frac{2}{1}, \pm \frac{3}{1}, \pm \frac{4}{1}, \pm \frac{6}{1}, \pm \frac{12}{1}$$

So possible roots are $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$

We can then plug these into $x^3 - 13x - 12$ to see if we get zero.

In this case

$$P(-1) = (-1)^3 - 13(-1) - 12 = 0$$

$$P(-3) = (-3)^3 - 13(-3) - 12 = 0$$

$$P(4) = (4)^3 - 13(4) - 12 = 0$$

and all other possibilities don't give us zero so we have some roots of

0, -1, -3, 4 Since the original equation

was $x^4 - 13x^2 - 12x$ has degree 4

this is all of the roots,

⑤

HW Example like HW 2 #46

Find all the rational zeros of the polynomial

$$P(x) = 4x^4 - 4x^3 - 4x^2 - 4x - 8$$

We apply the rational root theorem

$a_n = 4$, $a_0 = -8$ we will write the

factors of each

$$\begin{array}{c} \begin{array}{c} 4 \\ \hline \end{array} \left\{ \begin{array}{c} \pm 1 \\ \pm 2 \\ \pm 4 \end{array} \right\} \begin{array}{c} -8 \\ \hline \end{array} \left\{ \begin{array}{c} \pm 1 \\ \pm 2 \\ \pm 4 \\ \pm 8 \end{array} \right\} \end{array} \quad \begin{array}{c} q's \\ p's \end{array}$$

This gives the list
of p 's and q 's

To make p/q values

Listing all possible p/q

$$\frac{p}{q} = \left(\pm \frac{8}{1} \right), \left(\pm \frac{4}{1} \right), \left(\pm \frac{2}{1} \right), \left(\pm \frac{1}{1} \right)$$

$$\frac{p}{q} = \pm \frac{8}{2}, \pm \frac{4}{2}, \pm \frac{2}{2}, \left(\pm \frac{1}{2} \right)$$

$$\frac{p}{q} = \pm \frac{8}{4}, \pm \frac{4}{4}, \pm \frac{2}{4}, \left(\pm \frac{1}{4} \right)$$

we could try all
of these but from
the theorem p, q

should not have any
common factors other
than 1 so let's circle
those to narrow the
list

So now we have a much shorter list

$$\pm \frac{8}{1}, \pm \frac{4}{1}, \pm \frac{2}{1}, \pm \frac{1}{1}, \pm \frac{1}{2}$$

So $\pm 8, \pm 4, \pm 2, \pm 1, \pm \frac{1}{2}$ are possible rational roots. Of this list

$$P(-1) = 4(-1)^4 - 4(-1)^3 - 4(-1)^2 - 4(-1) - 8 = 0$$

$$P(2) = 4(2)^4 - 4(2)^3 - 4(2)^2 - 4(2) - 8 = 0$$

All others do not give $P(0) = 0$ and thus are not roots. Note that $P(x)$ was degree 4 so we are still missing roots they may be complex or irrational to find them we could use the fact that $(x - (-1)) = (x + 1)$ and $(x - 2)$ are factors of $P(x)$ and do polynomial long division to simplify the problem.

Complex roots

⑦

Suppose we want to find the x intercepts

of $f(x) = (x-1)^2 + 25$ we might begin

by setting the equation equal to zero and

Solving

$$(x-1)^2 + 25 = 0$$

$-25 \quad -25$

$$(x-1)^2 = -25$$

$$\sqrt{(x-1)^2} = \sqrt{-25} \quad \leftarrow \text{Here we get an imaginary}$$

$\neq$

$$(x-1) = \pm 5i$$

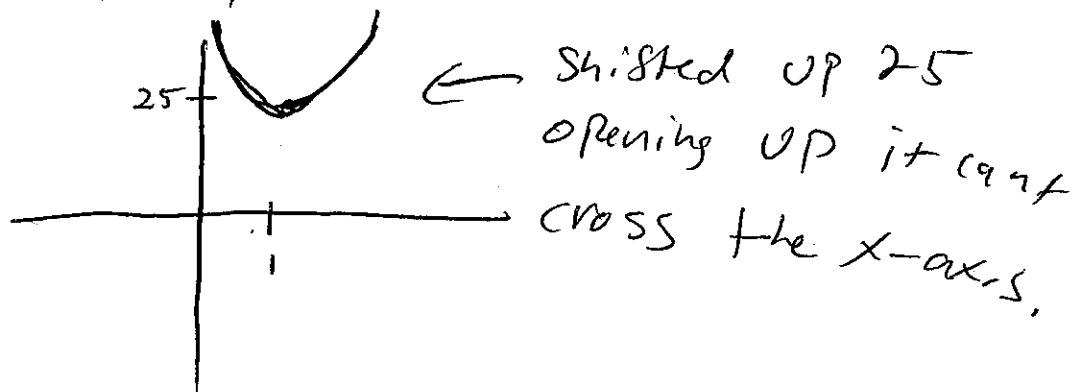
$+1 \quad +1$

$$x = 1 \pm 5i$$

So what do solutions of $x = 1 \pm 5i$ mean?

well here we have no x intercepts this

could be seen by sketching the graph



While complex roots can be annoying one nice feature is that they come in complex conjugate pairs. ⑧

Complex Zeros

Let $f(x)$ be a polynomial with real coefficients. If $a+bi$ is a zero of $f(x)$ then so is $a-bi$.

Recall the roots of ~~$x^2 + 10x + 25$~~

$(x-1)^2 + 25$ were $-1 \pm 5i$ so

$-1+5i$; $-1-5i$ and they are complex conjugates.

Descartes's Rule of Signs

Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_2 x^2 + a_1 x + a_0$ be a polynomial with real coefficients and $a_0 \neq 0$ then:

- ① The number of positive real zeros of f is either equal to the # of variations of signs of $f(x)$ or less than that by an even integer,

- ② The number of negative real zeros of f is either equal to the number of variations in sign of $f(-x)$ or less than that by an even integer.

HW Example Like #41 HW 2

Use Descartes's Rule of Signs to help list the possibilities for the nature of solutions

$$\text{to } 9x^5 + 4x^3 - x + 1 = 0$$

List possible numbers of positive real solutions

List possible numbers of negative real solutions

List possible numbers of non-real complex solutions.