

Practice Exam I

Math 1320 001 Summer 2014
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Julne 20th 2014

Name and Unid:

Solutions
by writing my name I swear by the honor code

Read all of the following information before starting the exam:

Note this is the practice exam ignore point values on problems there are extras but the exam will look similar

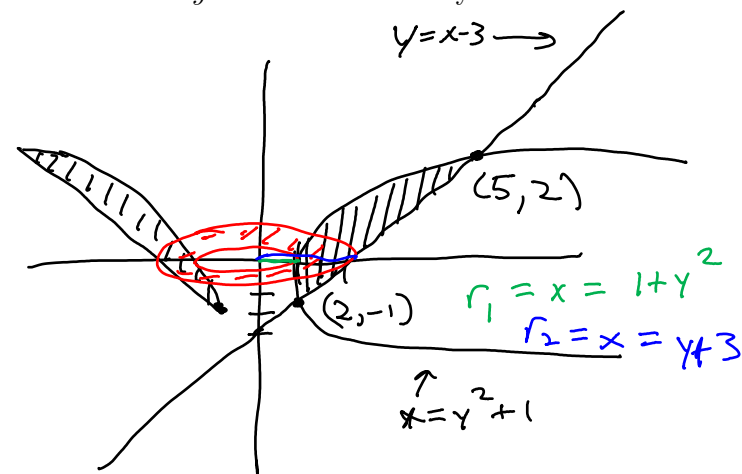
- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit. Sketch all relevant graphs and explain all relevant mathematics.
- Circle or otherwise indicate your final answers.
- You MAY NOT use a calculator or notes. You may use the formulas provided.
- This test has 11 problems and one bonus problem and is worth 100 points, plus some extra credit at the end. It is your responsibility to make sure that you have all of the pages!
- Good luck!

Good Luck!

Here are some Formulas and rules for you.

1. (5 points)

Find the volume of the solid obtained by rotating the region bound by the curves $x = 1 + y^2$, $y = x - 3$ about the y-axis



Integrate

Washers ^ along the y-axis

$$r_1 = 1 + y^2 \quad r_2 = y + 3$$

$$A(y) = \pi (r_2^2 - r_1^2)$$

$$A(y) = \pi ((y+3)^2 - (1+y^2)^2)$$

$$A(y) = \pi (y^2 + 6y + 9 - (1 + 2y^2 + y^4))$$

$$A(y) = \pi (-y^4 - y^2 + 6y + 8)$$

we need to find where the curves intersect $x = y^2 + 1$, $y = x - 3 \Rightarrow x = y + 3$

$$\Rightarrow y^2 + 1 = y + 3 \Rightarrow y^2 - y - 2 = 0$$

$$\Rightarrow (y+1)(y-2) = 0 \Rightarrow y = -1 \Rightarrow x = -1 + 3 = 2$$

$$y = 2 \Rightarrow x = 2 + 3 = 5$$

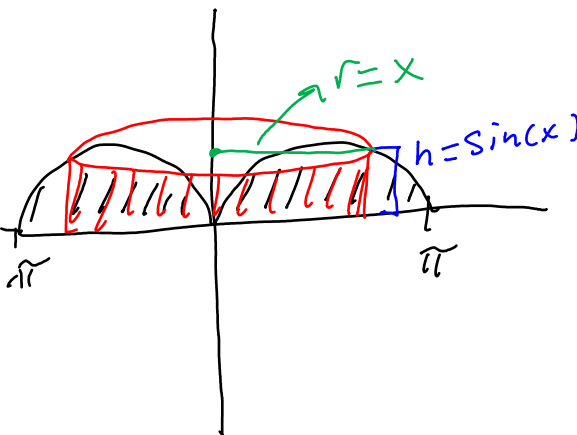
$$V = \int_{-1}^2 \pi (-y^4 - y^2 + 6y + 8) dy$$

$$\Rightarrow V = \pi \left(-\frac{y^5}{5} - \frac{y^3}{3} + \frac{6y^2}{2} + 8y \right) \Big|_{-1}^2$$

2. (10 points)

Find the volume of the solid obtained by rotating the region bound by the curves $x = \pi$, $y = \sin(x)$, $y = 0$ when rotated around the y-axis.

Cylindrical shell S Integrate along x-axis



$$A(x) = 2\pi r h = 2\pi x \sin(x)$$

$$A(x) = 2\pi x \sin(x)$$

$$V = \int_0^\pi 2\pi x \sin(x) dx$$

Integrate by Parts

$$u = x \quad dv = \sin(x)$$

$$du = dx \quad v = -\cos(x)$$

$$V = 2\pi \left(-x \cos(x) \right) \Big|_0^\pi - \int_0^\pi -\cos(x) dx$$

$$V = 2\pi \left(-x \cos(x) \right) \Big|_0^\pi + \sin(x) \Big|_0^\pi$$

$$V = 2\pi \left[-\pi \cos(\pi) - 0 \cos(0) + \sin(\pi) - \sin(0) \right]$$

$$V = 2\pi [\pi + 0] = 2\pi^2$$

$$\boxed{V = 2\pi^2}$$

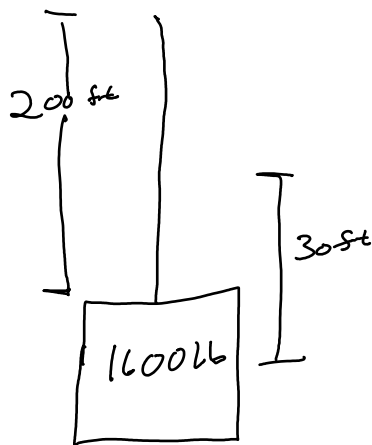
3. (10 points) The force on a particle moving along the x-axis from $x = 1$ to $x = 10$ is given by $F(x) = \ln(x)/x$ find the work done on the particle.

$$W = \int_a^b F(x) dx \Rightarrow W = \int_1^{10} \frac{\ln(x)}{x} dx \quad \begin{aligned} u &= \ln(x) \\ du &= \frac{1}{x} dx \end{aligned}$$

$$\Rightarrow W = \int u du = \frac{u^2}{2} = \frac{(\ln(x))^2}{2} \Big|_1^{10}$$

$$\Rightarrow W = \frac{(\ln(10))^2}{2} - \frac{(\ln(1))^2}{2} = \boxed{\frac{(\ln(10))^2}{2}}$$

4. (5 points) A 1600-lb elevator is suspended by a 200-ft cable that weighs 10 lb/ft. How much work is required to raise the elevator from the basement to the third floor, a distance of 30 ft?



We need the force as a function of x . There are 1600 lb for the elevator and 10 lb of rope for l ft of rope.

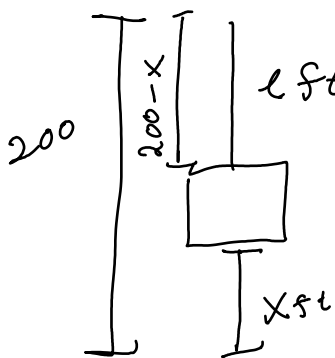
Suppose the elevator is at x ft. The total weight is $1600 + xl$.

Where l is the length of rope from the top to the elevator.

So $l = 200 - x$ so the force is

$$F(x) = 1600 + 10(200 - x) \text{ Thus}$$

$$\begin{aligned} W &= \int_0^{30} 1600 + 10(200 - x) dx = \int_0^{30} 1600 + 2000 - 10x dx \\ &= 1600x + 2000x - \frac{10x^2}{2} \Big|_0^{30} = 1600(30) + 2000(30) - \frac{10(30)^2}{2} \\ &= \boxed{103500 \text{ lb}\cdot\text{ft}} \end{aligned}$$



5. (10 points) You may not have known that I garden in the summer at home. One of my water pails has a small crack in the bottom and leaks as I fill it up from the hose. If the rate of water going into the pail is 0.2 L/sec and the leak rate is constant at .002 L/s Write a differential equation which describes the rate of change of the volume of water in the pail as I fill it up.

$$\boxed{\frac{dv}{dt} = 0.2 - .002}$$

$\uparrow$ rate in $\uparrow$ rate out

6. (10 points) Verify that $y = \cos(kx)$ is a solution to the differential equation $y'' + k^2y = 0$. ~~Not~~ don't try to solve just verify.

$$\begin{aligned}
 y &= \cos(kx) \\
 y' &= -k \sin(kx) \\
 y'' &= -k^2 \cos(kx)
 \end{aligned}$$

plus in to DE

$$\begin{aligned}
 y'' + k^2y &= 0 \\
 \downarrow \quad \quad \downarrow \\
 -k^2 \cos(kx) + k^2 \cos(kx) &= 0
 \end{aligned}$$

$$\hookrightarrow -k^2 \cos(kx) + k^2 \cos(kx) = 0 \quad \checkmark$$

since $y = \cos(kx)$ satisfies the DE it is a solution.

7. (10 points) Verify that $y = -t \cos(t) - t$ is a solution to the initial value problem $t \frac{dy}{dt} = y + t^2 \sin(t)$ with $y(\pi) = 0$. ~~Not~~ don't try to solve just verify.

$$y = -t \cos(t) - t$$

$$\frac{dy}{dt} = -\cos(t) + t \sin(t) - 1$$

check

$$t \frac{dy}{dt} = y + t^2 \sin(t)$$

$$t(-\cos(t) + t \sin(t) - 1) \stackrel{?}{=} -t \cos(t) - t + t^2 \sin(t)$$

$$\Rightarrow -t \cos(t) + t^2 \sin(t) - t = -t \cos(t) - t + t^2 \sin(t)$$

Both sides the same so $y = -t \cos(t) - t$ is a solution. $\checkmark$

check $y(\pi) = 0$

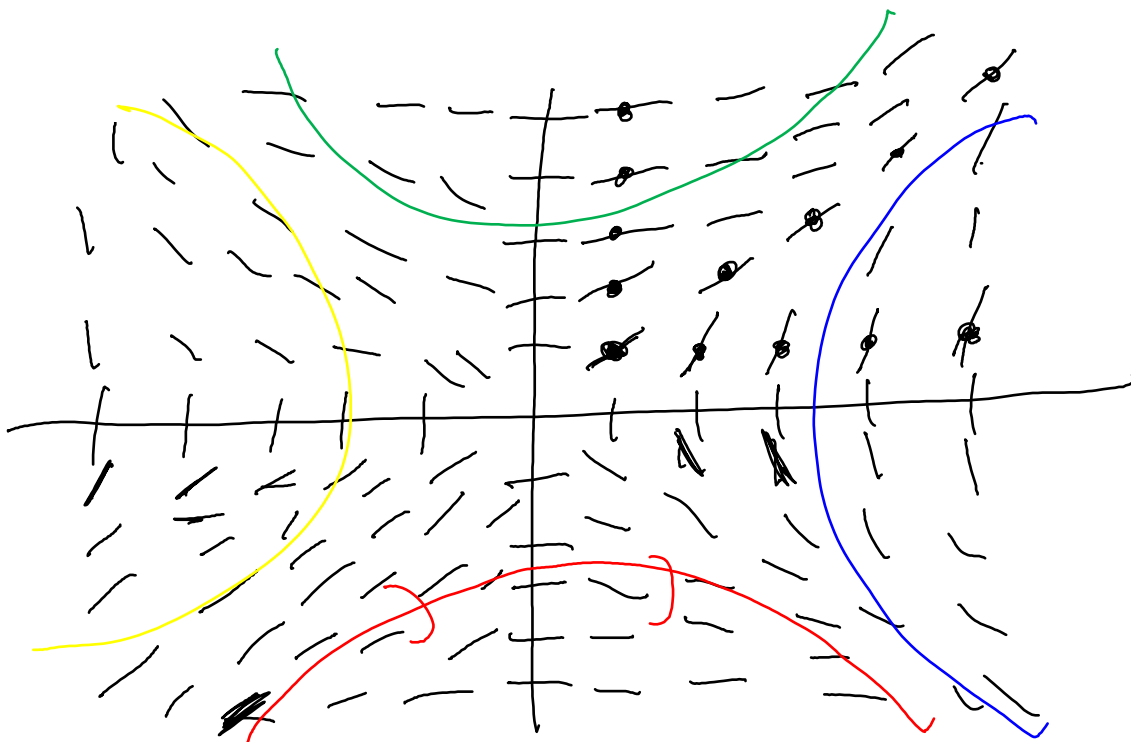
$$\begin{aligned}
 y(\pi) &= -\pi \cos(\pi) - \pi \\
 &= -\pi(-1) - \pi \\
 &= \pi - \pi = 0 \quad \checkmark
 \end{aligned}$$

satisfies both $y(\pi) = 0$ and $t \frac{dy}{dt} = y + t^2 \sin(t)$

8. (5 points) Sketch A direction field for the equation $y' = x/y$ and use it to sketch solution curves.

$y' = \frac{x}{y}$ plus in x, y points
say $(1, 2)$ then

$y' = \frac{1}{2}$ so the slope of the tangent
line at $(1, 2)$ is $1/2$ - - - -



The colored lines are
possible solution curves

9. (15 points)

a. (5 pts) Solve the initial value problem for the differential equation $\frac{dy}{dx} = x/y$ with $y(0)=1$

$$\frac{dy}{dx} = \frac{x}{y} \Rightarrow y dy = x dx \Rightarrow \int y dy = \int x dx$$

$$\Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + C \quad y(0)=1 \Rightarrow \frac{1^2}{2} = \frac{0^2}{2} + C$$

$$\Rightarrow C = 1/2$$

$$\Rightarrow \frac{y^2}{2} = \frac{x^2}{2} + \frac{1}{2} \Rightarrow y^2 = x^2 + 1 \quad \text{Hyperbola!}$$

$$\Rightarrow y^2 - x^2 = 1 \rightarrow \text{Hyperbola!}$$

b. (10 pts) Find the general solution to the differential equation $e^x h' = \sin(x)$

$$h' = \frac{dh}{dx} \Rightarrow e^x \frac{dh}{dx} = \sin(x)$$

$$\frac{dh}{dx} = \frac{\sin(x)}{e^x} = e^{-x} \sin(x) \Rightarrow dh = e^{-x} \sin(x) dx$$

$$\Rightarrow \int dh = \int e^{-x} \sin(x) dx \rightarrow \text{by parts} \quad \begin{array}{ll} u = \sin(x) & dv = e^{-x} \\ du = \cos(x) & v = -e^{-x} \end{array}$$

$$\Rightarrow \int e^{-x} \sin(x) dx = -e^{-x} \sin(x) - \int -e^{-x} \cos(x) dx \rightarrow \text{by parts} \quad \begin{array}{ll} u = \cos(x) & dv = -e^{-x} \\ du = -\sin(x) & v = e^{-x} \end{array}$$

$$\Rightarrow \int e^{-x} \sin(x) dx = -e^{-x} \sin(x) - (e^{-x} \cos(x) - \int -e^{-x} \sin(x) dx)$$

$$\Rightarrow \int e^{-x} \sin(x) dx = -e^{-x} \sin(x) - e^{-x} \cos(x) - \int e^{-x} \sin(x) dx$$

$$\Rightarrow \int e^{-x} \sin(x) dx = \boxed{\frac{1}{2} [-e^{-x} \sin(x) - e^{-x} \cos(x)] + C = h}$$

note we got the same integral again add it to both sides then solve for the integral

10. (10 points) Write the quadratic equation $f(x) = x^2 - 6x + 8$ in standard form.

woops from a 1050 Exam !

But you would just complete
the square.

11. (20 points) A population of bacteria grows exponentially and starts with 4 cells. After 20 minutes there are 8 cells.

a. (5 pts) Write down the equation that gives the population as a function of time

exponentially $\Rightarrow P(t) = P_0 e^{kt}$ here $P_0 = 4$

to find k we use the fact that after 20 min there are 8 cells so:

$$8 = 4e^{k(20)} \quad \text{Then solve for } k$$

$$\frac{8}{4} = e^{k(20)} \Rightarrow 2 = e^{20k} \Rightarrow \ln(2) = \ln(e^{20k})$$

$$\Rightarrow \ln(2) = 20k \Rightarrow k = \frac{\ln(2)}{20}$$

$$\Rightarrow \boxed{P(t) = 4e^{\frac{\ln(2)}{20}t}}$$

b. (5 pts) When will the population reach 100?

$$100 = 4e^{\frac{\ln(2)}{20}t} \quad \text{solve for } t$$

$$\frac{100}{4} = e^{\frac{\ln(2)}{20}t} \Rightarrow 25 = e^{\frac{\ln(2)}{20}t}$$

$$\Rightarrow \ln(25) = \ln(e^{\frac{\ln(2)}{20}t}) \Rightarrow \ln(25) = \frac{\ln(2)}{20}t$$

$$\Rightarrow \boxed{t = \frac{20 \ln(25)}{\ln(2)}}$$

12. (10 points)

Suppose that the rate a student learns new mathematics is proportional to the amount of mathematics they know already. Write a differential equation describing the rate that a student learns math. If you started in grade 1 with a math knowledge of 1 and after grade 2 you had a math knowledge of 4 find a function using the differential equation which gives the amount of math you know as a function of time. Consider grade 1 to 2 a time length of one year.

Suppose M is how math you know then

$$\boxed{\frac{dM}{dt} = kM} \quad \text{Solve the DE.}$$

$$\begin{aligned} \frac{dM}{M} &= k dt \Rightarrow \ln(M) = kt + C \Rightarrow e^{\ln(M)} = e^{kt+C} \\ &\Rightarrow M = e^{kt+C} \Rightarrow M = e^{kt} e^C \Rightarrow M = \tilde{C} e^{kt} \end{aligned}$$

Call grade 1 $t=0$ then $M(0) = 1$

$$\Rightarrow 1 = \tilde{C} e^{k(0)} \Rightarrow \tilde{C} = 1 \Rightarrow M = e^{kt}$$

after 1 year you have math knowledge

so $M(1) = 4$

$$\Rightarrow 4 = e^{k(1)} \Rightarrow 4 = e^k \Rightarrow \ln(4) = \ln(e^k)$$

$$\Rightarrow \ln(4) = k \Rightarrow \boxed{M(t) = e^{\ln(4)t}}$$

13. (15 points) Find the limits of the following sequences if they converge.

a. (5 pts) $a_n = \frac{3+5n}{n^4-n^3-n^2}$

$$\lim_{n \rightarrow \infty} \frac{3+5n}{n^4-n^3-n^2} = \frac{\infty}{\infty} \quad \text{apply L'Hopital's rule}$$

$$= \lim_{n \rightarrow \infty} \frac{5}{4n^3-3n^2-2n} = \boxed{0}$$

b. (5 pts) $a_n = \frac{e^n + e^{-2n}}{e^n - 1} = \frac{e^n + e^{-2n}}{e^n - 1}$

$$\lim_{n \rightarrow \infty} \frac{e^n + e^{-2n}}{e^n - 1} = \lim_{n \rightarrow \infty} \frac{e^{-n}}{e^{-n}} \left(\frac{e^n + e^{-2n}}{e^n - 1} \right) = \lim_{n \rightarrow \infty} \left(\frac{1 + e^{-3n}}{1 - e^{-n}} \right)$$

$$= \lim_{n \rightarrow \infty} \frac{1 + \cancel{e^{-3n} 0}}{1 - \cancel{e^{-n} 0}} = \frac{1}{1} = \boxed{1}$$

c. (5 pts) $a_n = n^n$

$$L = \lim_{n \rightarrow \infty} n^n \Rightarrow \ln(L) = \lim_{n \rightarrow \infty} \ln(n^n)$$

$$\Rightarrow \ln(L) = \lim_{n \rightarrow \infty} n \ln(n) = \infty \cdot \infty = \infty$$

$$\ln(L) = \infty \Rightarrow \boxed{L = \infty}$$

14. (10 points) Determine whether the given series is convergent or divergent. If it is convergent find its sum

a. (5 pts) $\sum_{n=1}^{\infty} \left(\frac{1}{\pi}\right)^n + \frac{1}{n} - \frac{1}{n+1}$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{\pi}\right)^n + \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$$

Geometric
with $r = \frac{1}{\pi} < 1$
starting at 1

Telescoping

$$\frac{1}{1 - \frac{1}{\pi}} - 1 + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \dots$$

$$= \frac{1}{1 - \frac{1}{\pi}} - 1 + 1 = \frac{1}{1 - \frac{1}{\pi}} = \sum_{n=1}^{\infty} \left(\frac{1}{\pi}\right)^n + \frac{1}{n} - \frac{1}{n+1}$$

$$\frac{1}{1 - \frac{1}{\pi}}$$

Note $\sum_{n=0}^{\infty} \left(\frac{1}{\pi}\right)^n = \frac{1}{1 - \frac{1}{\pi}}$

we have $\sum_{n=1}^{\infty} \left(\frac{1}{\pi}\right)^n$ which is missing

the $n=0$ term of the above series so we just subtract it from ours the $n=0$

term is 1 so $\sum_{n=1}^{\infty} \left(\frac{1}{\pi}\right)^n = \frac{1}{1 - \frac{1}{\pi}} - 1$

b. (5 pts) $\sum_{n=1}^{\infty} \frac{n^2 + 3n^5 - n^4}{n^3 - n^2 + n^5}$

$$\lim_{n \rightarrow \infty} \frac{n^2 + 3n^5 - n^4}{n^3 - n^2 + n^5} = \lim_{n \rightarrow \infty} \frac{3n^5 - n^4 + n^2}{n^5 + n^3 - n^2} = 3 \neq 0$$

By the limit test this series diverges.

c. (5 pts) $\sum_{n=1}^{\infty} \frac{\ln(n)}{n}$

$$\int_1^{\infty} \frac{\ln(x)}{x} dx$$

$$u = \ln(x)$$

$$du = \frac{1}{x} dx$$

$$= \int u du = \frac{u^2}{2} = \frac{(\ln(x))^2}{2} \Big|_1^{\infty} = \lim_{x \rightarrow \infty} \frac{(\ln(x))^2}{2} - \frac{(\ln(1))^2}{2}$$

$$= \infty - 0 = \infty \quad \boxed{\text{diverges}} \text{ by the integral test}$$

15. (10 points) Determine whether or not the given series is absolutely convergent, conditionally convergent, or divergent. $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+9}$

① Apply Alternating series / limit test

$$\lim_{n \rightarrow \infty} \frac{n}{n+9} = 1 \neq 0 \text{ so the series diverges}$$

16. (10 points) Determine how many terms you need to ensure that the partial sum of the series $\sum_{n=1}^{\infty} \frac{1}{n^4}$ is correct to within 0.0001

we have that $\int_{n+1}^{\infty} \frac{1}{x^4} \leq R_n \leq \int_n^{\infty} \frac{1}{x^4}$

We will find $\int_n^{\infty} \frac{1}{x^4}$ and find n such that it is less than 0.0001

$$\int_n^{\infty} \frac{1}{x^4} dx = \int_n^{\infty} x^{-4} dx = \left. \frac{x^{-3}}{-3} \right|_n^{\infty} = \left. -\frac{1}{3x^3} \right|_n^{\infty} = \lim_{x \rightarrow \infty} -\frac{1}{3x^3} - \left(-\frac{1}{3n^3} \right)$$

$$= \frac{1}{3n^3} \text{ so set } \frac{1}{3n^3} = 0.0001 \Rightarrow \frac{1}{3n^3} = \frac{1}{10000}$$

$$\Rightarrow 3n^3 = 10000 \Rightarrow n^3 = \frac{10000}{3} \Rightarrow n^3 = \frac{10000}{3}$$

$$\Rightarrow n = \left(\frac{10000}{3} \right)^{1/3} = 14.938$$

$$\Rightarrow n \geq 15$$

17. (10 points) Find the radius of convergence and interval of convergence for the following series.

a. (5 pts) $\sum_{n=1}^{\infty} \frac{(x-4)^n}{n^2}$ Apply the ratio test.

$$\lim_{n \rightarrow \infty} \left| \frac{(x-4)^{n+1}}{(n+1)^2} \cdot \frac{n^2}{(x-4)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{n^2}{(n+1)^2} (x-4) \right| = \left(\lim_{n \rightarrow \infty} \frac{n^2}{(n+1)^2} \right) |x-4|$$

$$= |x-4| \text{ to converge we want } |x-4| < 1 \leftarrow \boxed{\text{Radius of convergence} = 1}$$

For the interval of convergence $|x-4| < 1 \Rightarrow -1 < x-4 < 1$

$\Rightarrow -3 < x < 5$ check endpoints -3 and 5 .

for $x = -3$ $\sum \frac{(-3-4)^n}{n^2} = \sum \frac{(-7)^n}{n^2}$ for $x = 5$ $\sum \frac{(5-4)^n}{n^2} = \sum \frac{1}{n^2}$ Both converge by the p-test so the interval of convergence includes the endpoints.

$\Rightarrow$ Interval of convergence $= [-3, 5]$

b. (5 pts) $\sum_{n=1}^{\infty} \frac{(-1)^n x^n}{n!}$

$$\lim_{n \rightarrow \infty} \left| \frac{(-1)^{n+1} x^{n+1}}{(n+1)!} \cdot \frac{n!}{x^n (-1)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{x^{n+1}}{x^n} \cdot \frac{n!}{(n+1)!} \right| = \lim_{n \rightarrow \infty} \left| \frac{x}{n+1} \right| = 0$$

so here the limit is 0 for all x thus the radius of convergence is $R = \infty$ and interval of convergence is $(-\infty, \infty)$

In case you did not see it: $\frac{n!}{(n+1)!} = \frac{n!}{(n+1)(n)(n-1)(n-2)(n-3) \dots 1} = \frac{1}{n+1}$ In the steps above.

c. (5 pts) $\sum_{n=1}^{\infty} \frac{(-2)^n (x-1)^n}{n}$

$$\lim_{n \rightarrow \infty} \left| \frac{(-2)^{n+1} (x-1)^{n+1}}{n+1} \cdot \frac{n}{-2^n (x-1)^n} \right| = \lim_{n \rightarrow \infty} \left| \frac{2(x-1)^{n+1} n}{2^n (x-1)^n (n+1)} \right| = \lim_{n \rightarrow \infty} \left| 2(x-1) \frac{n}{n+1} \right|$$

$$= |2(x-1)| \lim_{n \rightarrow \infty} \left| \frac{n}{n+1} \right| = |2(x-1)| \text{ needs to be } < 1 \text{ for conv.}$$

$$\Rightarrow |2(x-1)| < 1 \Rightarrow |x-1| < \frac{1}{2} \leftarrow \text{radius of convergence } \boxed{R = \frac{1}{2}}$$

$$|x-1| < \frac{1}{2} \Rightarrow -\frac{1}{2} < x-1 < \frac{1}{2} \Rightarrow \frac{1}{2} < x < \frac{3}{2} \text{ check } x = \frac{1}{2} \text{ } x = \frac{3}{2}$$

for $x = \frac{1}{2}$

$$\sum \frac{(-2)^n (\frac{1}{2}-1)^n}{n} = \sum \frac{(-2)^n (-\frac{1}{2})^n}{n} = \sum \frac{1}{n} \text{ diverges}$$

for $x = \frac{3}{2}$

$$\sum \frac{(-2)^n (\frac{3}{2}-1)^n}{n} = \sum \frac{(-2)^n (\frac{1}{2})^n}{n} = \sum \frac{(-1)^n}{n} \lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

so converges by Alternating Series test.

Interval is: $(\frac{1}{2}, \frac{3}{2}]$

18. (10 points) Find a power series for the following functions using the geometric series.

$$\sum_{n=1}^{\infty} x^n = \frac{1}{1-x} \text{ when } |x| < 1$$

a. (5 pts) $f(x) = \frac{1}{1+x^2}$

plus in $-x^2$ for x

$$\frac{1}{1-x} = \sum x^n$$

$$\frac{1}{1-(-x^2)} = \sum (-x^2)^n$$

$$\boxed{\frac{1}{1+x^2} = \sum (-1)^n x^{2n}}$$

b. (5 pts) $f(x) = \frac{x}{(1-x^2)^2}$ ← plus in x^2 then take derivative

$$\frac{1}{1-x} = \sum x^n$$

$$\frac{1}{1-x^2} = \sum (x^2)^n = \sum x^{2n}$$

$$\frac{d}{dx} \frac{1}{1-x^2} = \frac{d}{dx} \sum x^{2n}$$

$$\frac{d}{dx} (1-x^2)^{-1} = \sum \frac{d}{dx} x^{2n}$$

$$\rightarrow -(1-x^2)^{-2} (2x) = \sum 2nx^{2n-1}$$

$$\frac{-2x}{(1-x^2)^2} = \sum 2nx^{2n-1} \quad \left\{ \begin{array}{l} \text{multiply both} \\ \text{sides by } -\frac{1}{2} \end{array} \right.$$

$$-\frac{1}{2} \left(\frac{-2x}{(1-x^2)^2} \right) = -\frac{1}{2} \sum 2nx^{2n-1}$$

$$\Rightarrow \boxed{\frac{x}{(1-x^2)^2} = \sum nx^{2n-1}}$$

c. (5 pts) $f(x) = \arctan(2x)$ ← plus in $-x^2$ Integrate then plus in $2x$

$$\frac{1}{1-x} = \sum x^n$$

$$\frac{1}{1-(-x^2)} = \sum (-x^2)^n$$

$$\frac{1}{1+x^2} = \sum (-1)^n x^{2n}$$

$$\int \frac{1}{1+x^2} dx = \sum \int (-1)^n x^{2n} dx$$

$$\rightarrow \arctan(x) = \sum (-1)^n \frac{x^{2n+1}}{2n+1}$$

$$\arctan(2x) = \sum (-1)^n \frac{(2x)^{2n+1}}{2n+1}$$

$$\Rightarrow \boxed{\arctan(2x) = \sum (-1)^n \frac{2^{2n+1} x^{2n+1}}{2n+1}}$$

Taylor
says

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

19. (10 points)

Find Taylor series out to 4 terms for the following functions at the given centers

a. (5 pts) $f(x) = \sqrt{x}$ centered at $x = 1$

here $a = 1$ want 4 terms so we need f, f', f'', f'''

$$\left. \begin{aligned} f(x) &= \sqrt{x} = x^{1/2} \\ f'(x) &= \frac{1}{2} x^{-1/2} \\ f''(x) &= -\frac{1}{4} x^{-3/2} \\ f'''(x) &= \frac{3}{8} x^{-5/2} \end{aligned} \right\} \begin{array}{l} \text{evaluate at} \\ x=1 \end{array} \rightarrow \begin{aligned} f(1) &= \sqrt{1} = 1 \\ f'(1) &= \frac{1}{2} (1)^{-1/2} = 1/2 \\ f''(1) &= -\frac{1}{4} (1)^{-3/2} = -1/4 \\ f'''(1) &= \frac{3}{8} (1)^{-5/2} = 3/8 \end{aligned}$$

$$\Rightarrow f(x) \approx f(1) + \frac{f'(1)(x-1)}{1!} + \frac{f''(1)(x-1)^2}{2!} + \frac{f'''(1)(x-1)^3}{3!}$$

$$\Rightarrow \boxed{f(x) \approx 1 + \frac{1}{2}(x-1) + \frac{-\frac{1}{4}(x-1)^2}{2} + \frac{\frac{3}{8}(x-1)^3}{6} = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{3}{48}(x-1)^3}$$

b. (5 pts) $f(x) = \cos(x)$ centered at $x = \pi$

$$\underline{\hspace{2cm}} \rightarrow a = \pi$$

$$\left. \begin{aligned} f(x) &= \cos(x) \\ f'(x) &= -\sin(x) \\ f''(x) &= -\cos(x) \\ f'''(x) &= \sin(x) \end{aligned} \right\} \begin{array}{l} \text{evaluate at center} \\ \text{at } x = \pi \end{array} \Rightarrow \begin{aligned} f(\pi) &= \cos(\pi) = -1 \\ f'(\pi) &= -\sin(\pi) = 0 \\ f''(\pi) &= -\cos(\pi) = -(-1) = 1 \\ f'''(\pi) &= \sin(\pi) = 0 \end{aligned}$$

$$f(x) \approx f(\pi) + \frac{f'(\pi)(x-\pi)}{1} + \frac{f''(\pi)(x-\pi)^2}{2} + \frac{f'''(\pi)(x-\pi)^3}{3!}$$

$$= -1 + 0(x-\pi) + \frac{1}{2}(x-\pi)^2 + \frac{0}{3!}(x-\pi)^3$$

$$= -1 + \frac{1}{2}(x-\pi)^2$$

20. (10 points)

Calculate the indefinite integral $\int \frac{\sin(x)}{x} dx$ using power series.

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \Rightarrow \frac{\sin(x)}{x} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \cdot \frac{1}{x}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$$

$$\Rightarrow \int \frac{\sin(x)}{x} = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \int x^{2n} dx$$

$$= \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot \frac{x^{2n+1}}{2n+1} + C \right)$$

If you do it term by term.....

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \Rightarrow \frac{\sin(x)}{x} = \frac{1}{x} \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} \right)$$

$$= \frac{x}{x} - \frac{x^3}{3!x} + \frac{x^5}{5!x} - \frac{x^7}{7!x} + \dots = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \dots = \sum \frac{(-1)^n x^{2n}}{(2n+1)!}$$

21. (5 points) (Bonus all or nothing) Prove using Taylor series the identity $e^{ix} = \cos(x) + i \sin(x)$. Then show that $e^{i\pi} = -1$

$$e^x = \sum \frac{x^n}{n!}$$

$$e^{ix} = \sum \frac{(ix)^n}{n!} = 1 + \frac{(ix)}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \frac{(ix)^6}{6!}$$

$$= 1 + ix + \frac{i^2 x^2}{2!} + \frac{i^3 x^3}{3!} + \frac{i^4 x^4}{4!} + \frac{i^5 x^5}{5!} + \frac{i^6 x^6}{6!} + \frac{i^7 x^7}{7!}$$

$$\text{note } i^2 = -1 \quad i^3 = -i \quad i^4 = 1 \quad i^5 = i \quad i^6 = -1 \quad i^7 = -i$$

$$= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} \quad \leftarrow \begin{array}{l} \text{group real} \\ \text{and imaginary} \\ \text{parts} \end{array}$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \right) + \left(ix - \frac{ix^3}{3!} + \frac{ix^5}{5!} - \frac{ix^7}{7!} + \dots \right)$$

$$= \underbrace{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \right)}_{=\cos(x)} + i \underbrace{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \right)}_{=\sin(x)}$$

$$\Rightarrow e^{ix} = \cos(x) + i \sin(x)$$

$$\text{Thus } e^{i\pi} = \cos(\pi) + i \sin(\pi) = -1 + 0 = -1$$

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