

Practice Exam II

Math 1320 001 Summer 2014
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July 25th 2014

Name and Unid:

Solutions

by writing my name I swear by the honor code

Read all of the following information before starting the exam:

Note this is the practice exam ignore point values on problems there are extras but the exam will look similar

- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit. Sketch all relevant graphs and explain all relevant mathematics.
- Circle or otherwise indicate your final answers.
- You MAY NOT use a calculator or notes. You may use the formulas provided.
- This test has 11 problems and one bonus problem and is worth 100 points, plus some extra credit at the end. It is your responsibility to make sure that you have all of the pages!
- Good luck!

Good Luck!

Here are some Formulas and rules for you.

1. (5 points)

Find the distance between the points $(1, 4, -6)$ and $(2, 3, 7)$

Here we can use the distance formula

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$\begin{aligned}\Rightarrow d &= \sqrt{(2-1)^2 + (3-4)^2 + (7-(-6))^2} \\ &= \sqrt{1^2 + 1^2 + (13)^2} \\ &= \sqrt{1 + 1 + 169} \\ &= \sqrt{171}\end{aligned}$$

2. (10 points)

Show that the equation represents a ~~sphere~~ ^{ellipsoid}, find its center and radius. $2x^2 + y^2 + z^2 + 8x - 6y + 2z = 0$

Here we will complete the square for x, y, z

$$2x^2 + y^2 + z^2 + 8x - 6y + 2z = 0$$

$$2x^2 + 8x + y^2 - 6y + z^2 + 2z = 0$$

$$2(x^2 + 4x) + y^2 - 6y + z^2 + 2z = 0$$

$$2(\underbrace{x^2 + 4x + 4}_{-4}) + (\underbrace{y^2 - 6y + 9}_{-9}) + (\underbrace{z^2 + 2z + 1}_{-1}) = 0$$

$$2((x+2)^2 - 4) + ((y-3)^2 - 9) + ((z+1)^2 - 1) = 0$$

$$2(x+2)^2 - 8 + (y-3)^2 - 9 + (z+1)^2 - 1 = 0$$

$+8 \quad +9 \quad +1 \quad +8+9+1$

$$2(x+2)^2 + (y-3)^2 + (z+1)^2 = 18$$

$$\boxed{\frac{(x+2)^2}{9} + \frac{(y-3)^2}{18} + \frac{(z+1)^2}{18} = 1}$$

ellipsoid centered at $(-2, 3, 1)$

for $x^2 + 4x \rightarrow \frac{4}{2} = 2$
 $2^2 = 4$

for $y^2 - 6y \rightarrow \frac{6}{2} = 3$
 $3^2 = 9$

for $z^2 + 2z \rightarrow \frac{2}{2} = 1$
 $1^2 = 1$

3. (10 points) Find $\vec{a} + \vec{b}$, $\vec{a} - \vec{b}$, $|\vec{a}|$, $|\vec{a} - \vec{b}|$ for the vectors $\vec{a} = \langle 1, 4, -2 \rangle$, $\vec{b} = \langle 2, 5, -1 \rangle$

$$\vec{a} + \vec{b} = \langle 1, 4, -2 \rangle + \langle 2, 5, -1 \rangle = \langle 1+2, 4+5, -2-1 \rangle \\ = \langle 3, 9, -3 \rangle$$

$$\vec{a} - \vec{b} = \langle 1, 4, -2 \rangle - \langle 2, 5, -1 \rangle = \langle 1-2, 4-5, -2+1 \rangle \\ = \langle -1, -1, -1 \rangle$$

$$|\vec{a}| = \sqrt{1^2 + 4^2 + (-2)^2} = \sqrt{1+16+4} = \sqrt{21}$$

$$|\vec{a} - \vec{b}| = |\langle -1, -1, -1 \rangle| = \sqrt{(-1)^2 + (-1)^2 + (-1)^2} = \sqrt{3}$$

4. (5 points) Find a unit vector in the same direction as $\vec{v} = \langle -2, 4, 2 \rangle$. Also find a vector in the same direction but with length 6.

$$\text{unit vec} = \frac{\vec{v}}{|\vec{v}|}, \quad |\vec{v}| = \sqrt{(-2)^2 + 4^2 + 2^2} = \sqrt{4+16+4}$$

$$= \sqrt{24} \Rightarrow \vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{1}{\sqrt{24}} \langle -2, 4, 2 \rangle$$

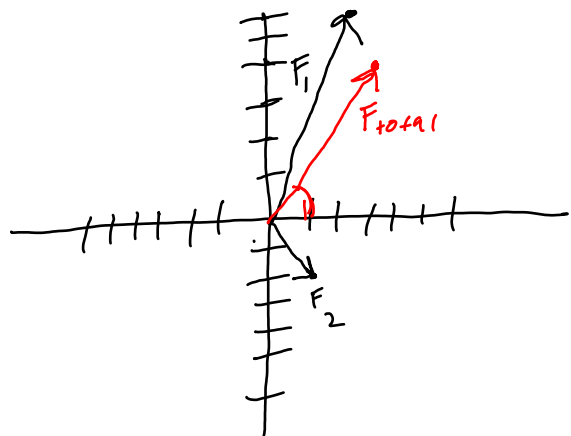
$$= \left\langle \frac{-2}{\sqrt{24}}, \frac{4}{\sqrt{24}}, \frac{2}{\sqrt{24}} \right\rangle$$

Vector in the same direction with length 6

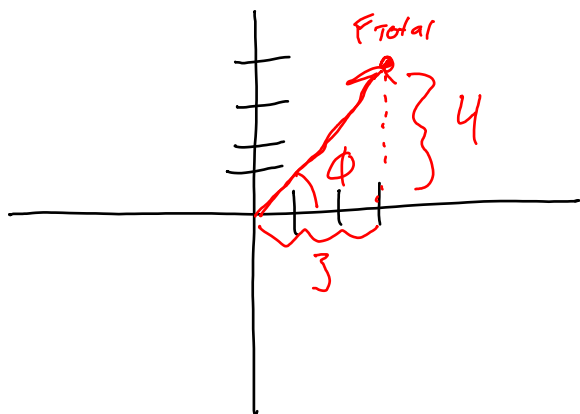
our unit vector is already in the direction of $\vec{v}$ with length 1 so we can simply scale the unit vector by 6...

$$6 \left\langle \frac{-2}{\sqrt{24}}, \frac{4}{\sqrt{24}}, \frac{2}{\sqrt{24}} \right\rangle = \left\langle \frac{-12}{\sqrt{24}}, \frac{24}{\sqrt{24}}, \frac{12}{\sqrt{24}} \right\rangle$$

5. (10 points) Two forces are acting on a particle in the x-y plane given by: $F_1 = \langle 2, 6 \rangle$ and $F_2 = \langle 1, -2 \rangle$. Draw these two force vectors. What is the angle between the vectors? Find the total force vector and determine what angle will the particle move relative to the x-axis.



$$\begin{aligned} \text{Total Force} &= \vec{F}_1 + \vec{F}_2 \\ &= \langle 2, 6 \rangle + \langle 1, -2 \rangle \\ &= \langle 3, 4 \rangle = \vec{F}_{\text{total}} \end{aligned}$$



Angle between: $\vec{F}_1 \cdot \vec{F}_2 = |\vec{F}_1| |\vec{F}_2| \cos \theta$

So $\cos \theta = \frac{\vec{F}_1 \cdot \vec{F}_2}{|\vec{F}_1| |\vec{F}_2|}$

$$\vec{F}_1 \cdot \vec{F}_2 = \langle 2, 6 \rangle \cdot \langle 1, -2 \rangle = 2(1) + 6(-2) = -10$$

$$|\vec{F}_1| = \sqrt{2^2 + 6^2} = \sqrt{40} \quad |\vec{F}_2| = \sqrt{1^2 + 2^2} = \sqrt{5}$$

$$\begin{aligned} \Rightarrow \cos \theta &= \frac{-10}{\sqrt{40} \sqrt{5}} = \frac{-10}{\sqrt{40(5)}} = \frac{-10}{\sqrt{200}} \\ &= \frac{-10}{10\sqrt{2}} = -\frac{1}{\sqrt{2}} \end{aligned}$$

$$\Rightarrow \cos \theta = -\frac{1}{\sqrt{2}} \Rightarrow \theta = \cos^{-1}(-\frac{1}{\sqrt{2}})$$

$$\Rightarrow \boxed{\theta = \frac{3\pi}{4}} \text{ or } \boxed{\theta = 135^\circ}$$

The particle will move at angle ϕ relative to the x-axis and $\cos \phi = \frac{3}{4} \Rightarrow \boxed{\phi = \cos^{-1}(\frac{3}{4})}$

This would be fine.

6. (10 points) Determine which of the following pairs of vectors are orthogonal.

- $\langle 1, 4, 6 \rangle, \langle -1, -4, 6 \rangle \rightarrow \langle 1, 4, 6 \rangle \cdot \langle -1, -4, 6 \rangle = -1 - 16 - 36 \neq 0$
- $\langle 1, 4 \rangle, \langle 4, -1 \rangle \rightarrow \langle 1, 4 \rangle \cdot \langle 4, -1 \rangle = 4 - 4 = 0$ orthogonal!
- $\langle 1, -2, 1 \rangle, \langle -1, -1, -1 \rangle \rightarrow \langle 1, -2, 1 \rangle \cdot \langle -1, -1, -1 \rangle = -1 + 2 - 1 = 0$ orthogonal!
- $\langle 1, 0 \rangle, \langle -1, 0 \rangle \rightarrow \langle 1, 0 \rangle \cdot \langle -1, 0 \rangle = -1 + 0 \neq 0$

Recall that $\vec{a}$ is orthogonal to $\vec{b}$ if $\vec{a} \cdot \vec{b} = 0$

7. (5 points) Find the cross product between $\vec{a} = \langle 6, 0, -2 \rangle$ and $\vec{b} = \langle 0, 4, 2 \rangle$ and verify it is orthogonal to both $\vec{a}$ and $\vec{b}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} +\hat{i} & -\hat{j} & +\hat{k} \\ 6 & 0 & -2 \\ 0 & 4 & 2 \end{vmatrix} = +\hat{i} \begin{vmatrix} 0 & -2 \\ 4 & 2 \end{vmatrix} - \hat{j} \begin{vmatrix} 6 & -2 \\ 0 & 2 \end{vmatrix} + \hat{k} \begin{vmatrix} 6 & 0 \\ 0 & 4 \end{vmatrix}$$

$$= \hat{i} \begin{vmatrix} 0 & -2 \\ 4 & 2 \end{vmatrix} - \hat{j} \begin{vmatrix} 6 & -2 \\ 0 & 2 \end{vmatrix} + \hat{k} \begin{vmatrix} 6 & 0 \\ 0 & 4 \end{vmatrix}$$

$$= \hat{i} (0(2) - (4)(-2)) - \hat{j} (6(2) - (0)(-2)) + \hat{k} (6(4) - 0(0))$$

$$= 8\hat{i} - 12\hat{j} + 24\hat{k} = \boxed{\langle 8, -12, 24 \rangle}$$

check orthogonality with dot products

$$\langle 8, -12, 24 \rangle \cdot \langle 6, 0, -2 \rangle = 8(6) - 12(0) - 2(24) = 48 - 48$$

$$= 0 \checkmark$$

$$\langle 8, -12, 24 \rangle \cdot \langle 0, 4, 2 \rangle = 8(0) + (-12)(4) + (24)(2)$$

$$= -48 + 48 = 0 \checkmark$$

8. (15 points)

a. (5 pts) Find the symmetric, parametric, and vector equations of the line between the points $(6, 1, 2)$ and $(2, 3, 5)$

Steps:

① make a vector between the points by subtracting
This is then the vector
 $\vec{v} = \langle a, b, c \rangle$ the direction vector

② Pick either point to be (x_0, y_0, z_0)

③ Plug into:

symmetric eqns

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

parametric:

$$x = x_0 + at \quad y = y_0 + bt \quad z = z_0 + ct$$

vector:

$$\vec{r}(t) = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$$

$$(6, 1, 2) - (2, 3, 5) = \langle 6-2, 1-3, 2-5 \rangle = \langle 4, -2, -3 \rangle$$

$$\text{so } \vec{v} = \langle 4, -2, -3 \rangle = \langle a, b, c \rangle \Rightarrow \begin{matrix} a=4 \\ b=-2 \\ c=-3 \end{matrix}$$

choose either point i'll pick $(2, 3, 5)$

so:

symmetric:

$$\frac{x-2}{4} = \frac{y-3}{-2} = \frac{z-5}{-3}$$

param:

$$x = 2 + 4t \quad y = 3 - 2t \quad z = 5 - 3t$$

vector:

$$\vec{r}(t) = \langle 2 + 4t, 3 - 2t, 5 - 3t \rangle$$

b. (10 pts) Find the equation of the plane that contains the points $(1, 2, 4)$, $(4, -2, 3)$ and $(5, 2, 1)$

Steps:

① Find two vectors in the plane using the given points.

② Take the crossproduct of the two vectors in the plane to get a normal vector: $\vec{n} = \langle a, b, c \rangle$

③ Pick any of the points in the plane to be (x_0, y_0, z_0)

④ Plug into:

$$a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

Two vects in plane

$$(1, 2, 4) - (4, -2, 3) = \langle 1-4, 2-(-2), 4-3 \rangle$$

$$= \langle -3, 4, 1 \rangle$$

and

$$(1, 2, 4) - (5, 2, 1) = \langle 1-5, 2-2, 4-1 \rangle$$

$$= \langle -4, 0, 3 \rangle$$

Cross prod:

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -3 & 4 & 1 \\ -4 & 0 & 3 \end{vmatrix} = \hat{i} \begin{vmatrix} 4 & 1 \\ 0 & 3 \end{vmatrix} - \hat{j} \begin{vmatrix} -3 & 1 \\ -4 & 3 \end{vmatrix} + \hat{k} \begin{vmatrix} -3 & 4 \\ -4 & 0 \end{vmatrix}$$

$$= \hat{i}(12-0) - \hat{j}(-9+4) + \hat{k}(0-(-16)) = \langle 12, -5, 16 \rangle$$

choose $(x_0, y_0, z_0) = (1, 2, 4)$ Then the plane is

$$12(x-1) - 5(y-2) + 16(z-4) = 0$$

9. (10 points) Find the distance from the point $P(4, 1, -2)$ to the line $\vec{r}(t) = \langle 1+t, 3-2t, 4-3t \rangle$

Steps:

- ① we want the shortest distance from the point to the line



notice that the line segment from P to Q is orthogonal to the line L and Q is on the line. we don't know where Q is but suppose the point Q corresponds to time t_Q then

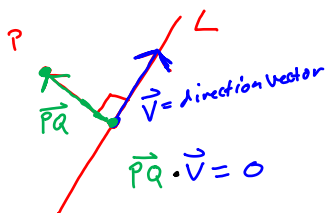
$Q = (1+t_Q, 3-2t_Q, 4-3t_Q)$ and the vector from Q to P is: $P-Q$

$$P-Q = (4, 1, -2) - (1+t_Q, 3-2t_Q, 4-3t_Q)$$

$$= \langle 4-(1+t_Q), 1-(3-2t_Q), -2-(4-3t_Q) \rangle$$

$$= \langle 3-t_Q, -2+2t_Q, -6+3t_Q \rangle$$

- ② since this vector needs to be orthogonal to the direction vector of the line we can take their dot products, set it equal to 0 then solve for t_Q then we have the vector $\vec{PQ}$ whose magnitude is the distance we want to find.



Note $\vec{V}$ comes from the equation of the line

$$\langle 1+t, 3-2t, 4-3t \rangle = \vec{r}(t)$$

$$\Rightarrow \vec{V} = \langle 1, -2, -3 \rangle$$

Let Q be the point on the line closest to $(4, 1, -2)$ and

corresponds to time t_Q then

$$\vec{PQ} = (4, 1, -2) - (1+t_Q, 3-2t_Q, 4-3t_Q)$$

$$= \langle 3-t_Q, -2+2t_Q, -6+3t_Q \rangle$$

The direction vector for our line is

$$\vec{V} = \langle 1, -2, -3 \rangle \text{ then since } Q \text{ is the}$$

point on $\vec{r}(t)$ closest to P it must be that:

$$\vec{PQ} \cdot \vec{V} = 0$$

$$\Rightarrow \langle 3-t_Q, -2+2t_Q, -6+3t_Q \rangle \cdot \langle 1, -2, -3 \rangle = 0$$

$$\Rightarrow 3-t_Q + (-2)(-2+2t_Q) + (-3)(-6+3t_Q) = 0$$

$$\Rightarrow 3-t_Q + 4 - 4t_Q + 18 - 9t_Q = 0$$

$$\Rightarrow 35 - 14t_Q = 0$$

$$\Rightarrow t_Q = \frac{35}{14} = \frac{5}{2}$$

$$\Rightarrow \vec{PQ} = \langle 3-\frac{5}{2}, -2-2(\frac{5}{2}), -6+3(\frac{5}{2}) \rangle$$

$$= \langle \frac{1}{2}, -7, \frac{3}{2} \rangle$$

Then $|\vec{PQ}| = \text{distance we want so}$

$$|\vec{PQ}| = \sqrt{(\frac{1}{2})^2 + 7^2 + (\frac{3}{2})^2} = \sqrt{\frac{1}{4} + 49 + \frac{9}{4}} = \sqrt{\frac{4}{4} + 49}$$

$$= \sqrt{1+49} = \boxed{\sqrt{50}}$$

10. (20 points) Consider the function $f(x, y) = \ln(\sqrt{4 - x^2 + y^2})$

a. (5 pts) Find the domain of $f(x, y)$

we know $\ln$ can't have 0 or negative numbers
plugged in thus we require

$$\sqrt{4 - x^2 + y^2} > 0$$

$$\Rightarrow 4 - x^2 + y^2 > 0$$

$$\Rightarrow x^2 - y^2 > 4$$

$$\Rightarrow \left[\frac{x^2}{4} - \frac{y^2}{4} > 1 \right]$$

or

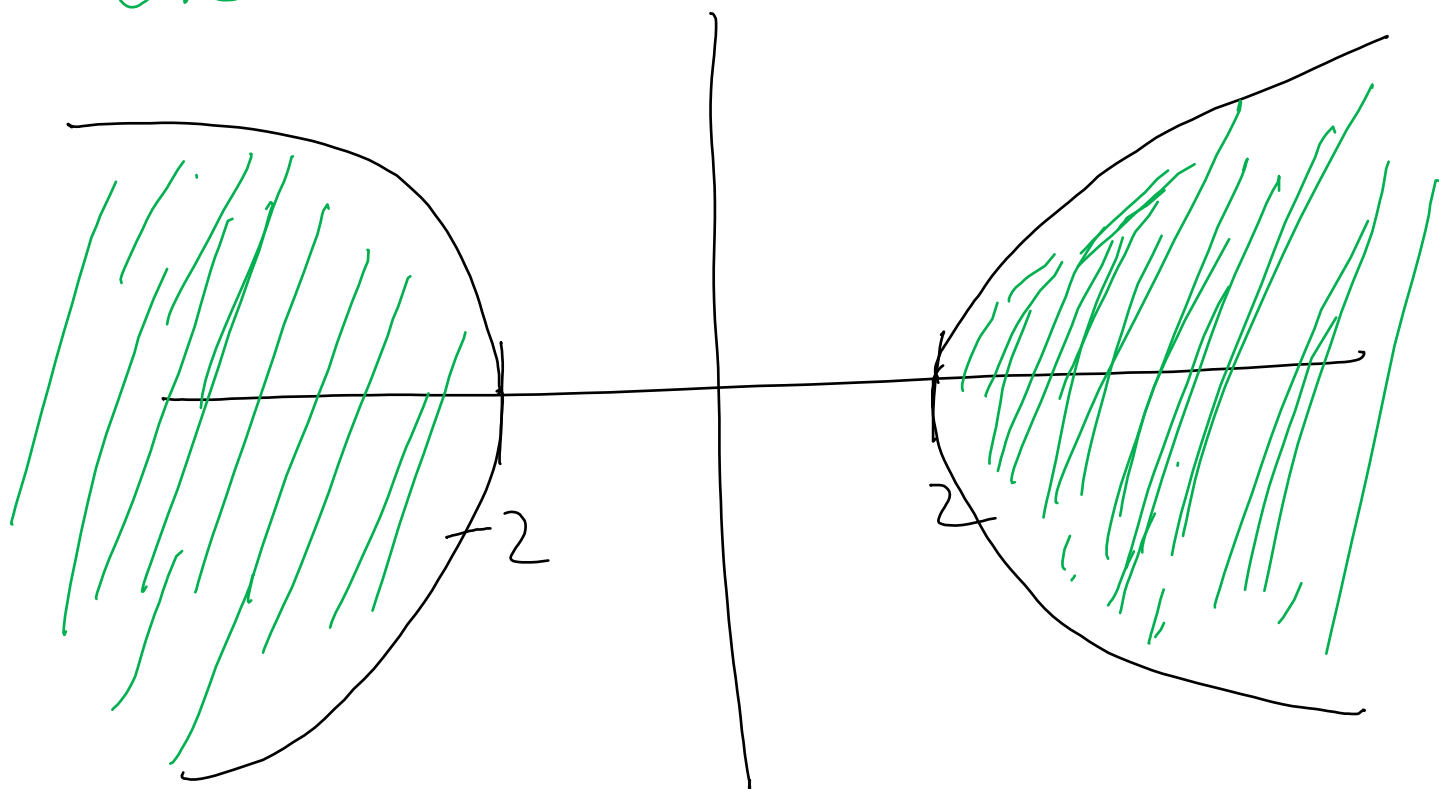
$$\{ (x, y) \mid 4 - x^2 + y^2 > 0 \}$$

b. (5 pts) Sketch the domain of $f(x, y)$

$\frac{x^2}{4} - \frac{y^2}{4} = 1$ is a Hyperbola want

$$\left[\frac{x^2}{4} - \frac{y^2}{4} > 1 \right]$$

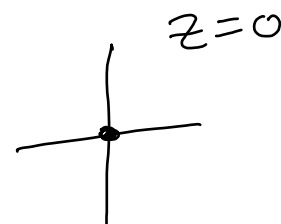
Green is domain



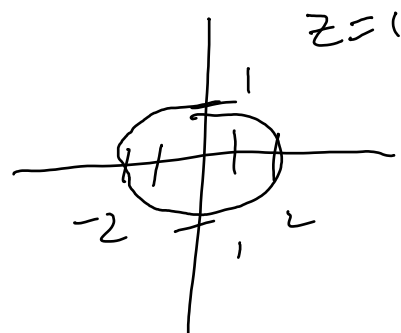
11. (10 points) Consider the function $f(x, y) = \frac{x^2}{4} + y^2$ Sketch level sets for $z = 0, 1, 2, 3$ Use those to sketch a graph of $f(x, y)$

level sets:

$$z=0 \Rightarrow 0 = \frac{x^2}{4} + y^2 \Rightarrow x=y=0$$

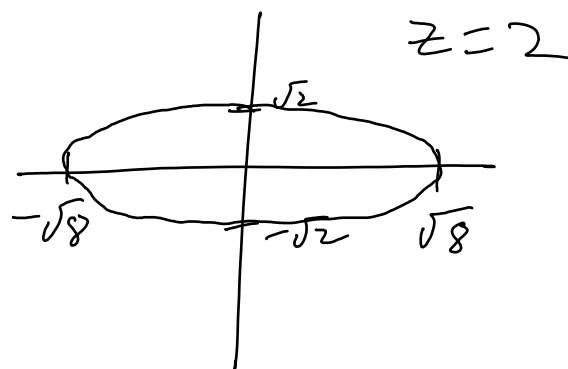


$$z=1 \Rightarrow 1 = \frac{x^2}{4} + y^2 \Rightarrow \text{ellipse}$$



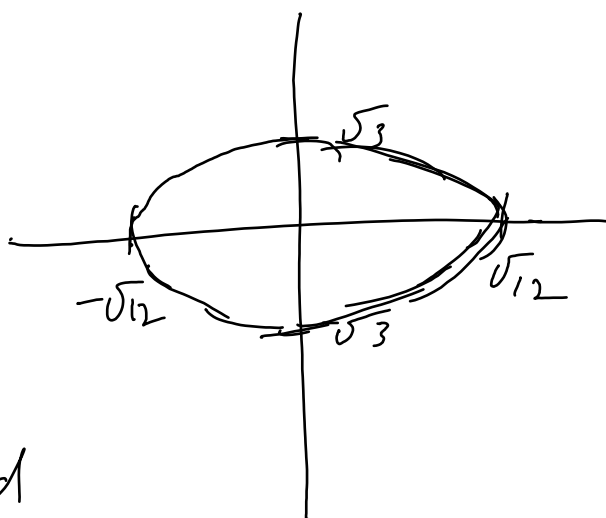
$$z=2 \Rightarrow 2 = \frac{x^2}{4} + y^2$$

$$\Rightarrow 1 = \frac{x^2}{8} + \frac{y^2}{2} \Rightarrow \text{ellipse}$$

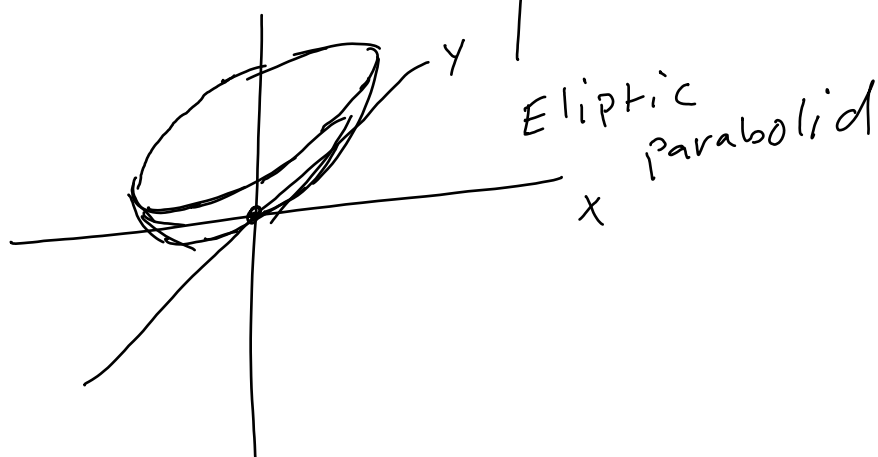


$$z=3 \Rightarrow 3 = \frac{x^2}{4} + y^2$$

$$\Rightarrow 1 = \frac{x^2}{12} + \frac{y^2}{3} \Rightarrow \text{ellipse}$$



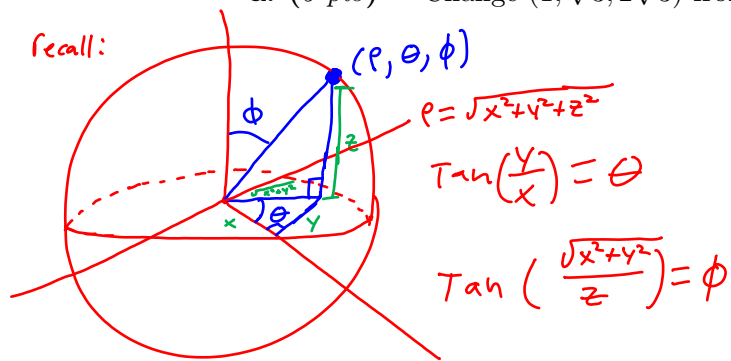
All together



12. (15 points) Do the following coordinate changes and plots

a. (5 pts) Change $(1, \sqrt{3}, 2\sqrt{3})$ from Rectangular to spherical coordinates.

Recall:



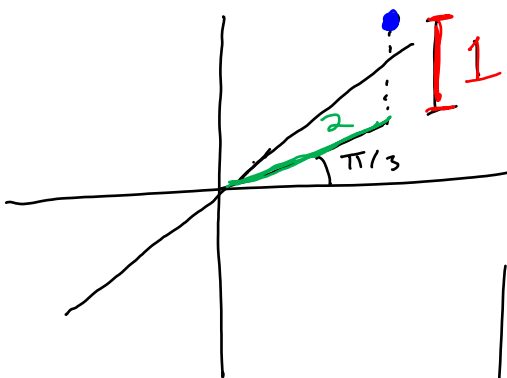
$$\rho = \sqrt{1^2 + (\sqrt{3})^2 + (2\sqrt{3})^2} = \sqrt{1 + 3 + 4(3)} = \sqrt{16} = 4$$

$$\tan(\theta) = \left(\frac{\sqrt{3}}{1}\right) \Rightarrow \theta = \pi/3$$

$$\tan(\phi) = \left(\frac{\sqrt{1^2 + \sqrt{3}^2}}{2\sqrt{3}}\right) = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}} \Rightarrow \phi = \pi/6$$

b. (5 pts) Plot the point given in cylindrical coordinates as $(2, \pi/3, 1)$ Then convert it to rectangular coordinates.

$$r = 2 \quad \theta = \pi/3 \quad z = 1$$



Recall

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$z = z$$

$$x = 2 \cos(\pi/3)$$

$$= 2(1/2) = 1$$

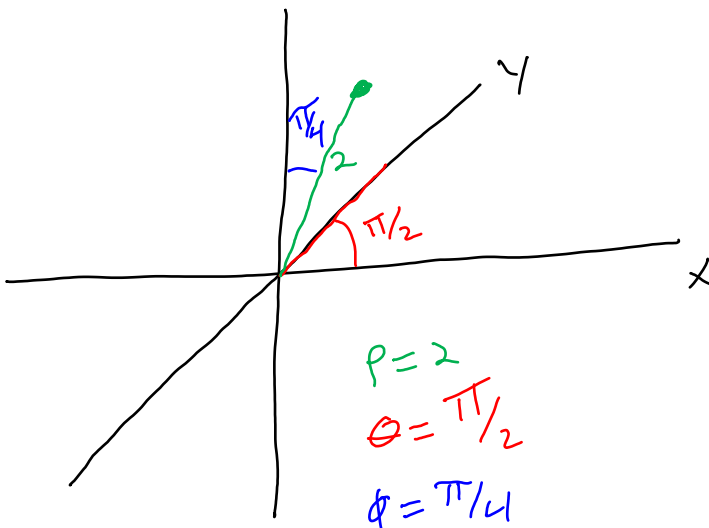
$$y = 2 \sin(\pi/3) = \sqrt{3}$$

$$z = 1$$

$$(2, \pi/3, 1) \rightarrow (1, \sqrt{3}, 1)$$

$\begin{matrix} r & \theta & z \\ x & y & z \end{matrix}$

c. (5 pts) Plot the point in spherical coordinates given by $(2, \pi/2, \pi/4)$ Then convert it to rectangular coordinates.



Recall

$$x = \rho \sin \phi \cos \theta$$

$$y = \rho \sin \phi \sin \theta$$

$$z = \rho \cos \phi$$

$$x = 2 \sin \pi/4 \cos \pi/2 = 2(\frac{\sqrt{2}}{2})(0) = 0$$

$$y = 2 \sin \pi/4 \sin \pi/2 = 2 \frac{\sqrt{2}}{2}(1) = \sqrt{2}$$

$$z = 2 \cos \pi/4 = 2 \frac{\sqrt{2}}{2} = \sqrt{2}$$

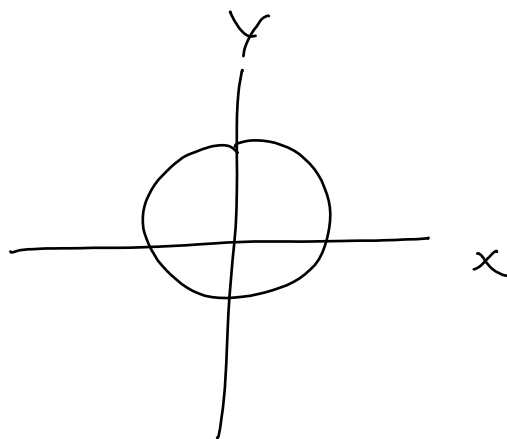
13. (10 points) Make a sketch of the curve given by the vector equation $\vec{r}(t) = \langle 2\cos(t), 2\sin(t), 2t \rangle$

we can Ignore parts of the vector and make 2-d sketches to help us.

Ignore the z part...

$$\langle 2\cos(t), 2\sin(t) \rangle$$

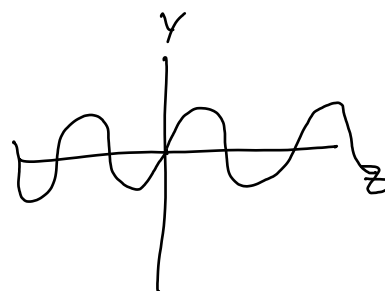
↑
a circle of radius
2



Ignore x part

$$\langle 2\sin(t), 2t \rangle$$

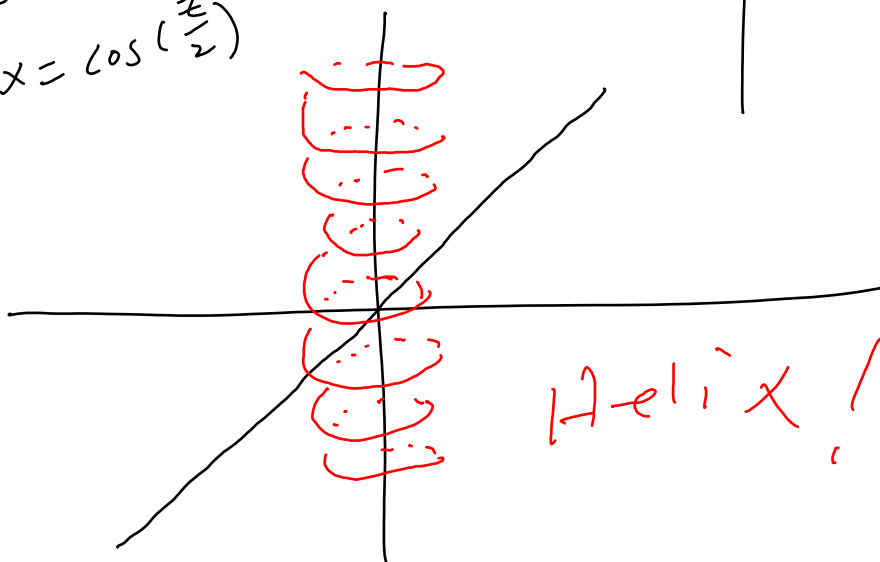
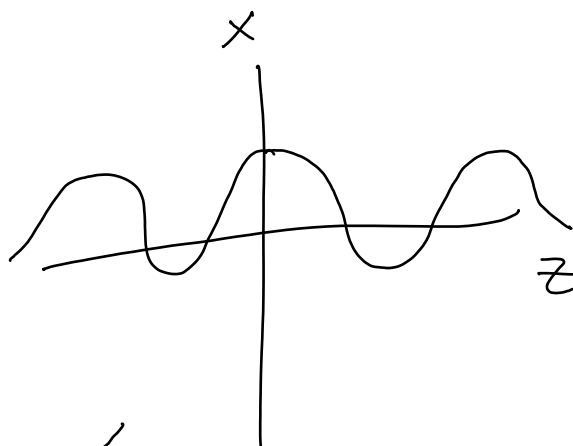
suppose $z = 2t$
then $y = 2\sin(\frac{z}{2})$



Ignore y part

$$\langle 2\cos(t), 2t \rangle$$

Let $z = 2t$
then $x = 2\cos(\frac{z}{2})$



Helix!

14. (10 points) Find the velocity, acceleration and speed of a particle whose position is given by $\vec{r}(t) = \langle 1+t^2, t^2, \sin(\pi t) \rangle$ at $t = 1$

$$\vec{v}(t) = \frac{d}{dt} \vec{r}(t) = \frac{d}{dt} \langle 1+t^2, t^2, \sin(\pi t) \rangle = \langle 2t, 2t, \pi \cos(\pi t) \rangle$$

$$\vec{v}(1) = \langle 2(1), 2(1), \pi \cos(\pi(1)) \rangle = \boxed{\langle 2, 2, -\pi \rangle}$$

$$\vec{a}(t) = \frac{d}{dt} \vec{v}(t) = \frac{d}{dt} \langle 2t, 2t, \pi \cos(\pi t) \rangle = \langle 2, 2, -\pi^2 \sin(\pi t) \rangle$$

$$\vec{a}(1) = \langle 2, 2, -\pi^2 \sin(\pi(1)) \rangle = \boxed{\langle 2, 2, 0 \rangle}$$

The speed = $|\vec{v}(t)|$ at $t=1$ $|\vec{v}(1)|$

$$|v(1)| = \sqrt{2^2 + 2^2 + \pi^2} = \sqrt{4 + 4 + \pi^2} = \boxed{\sqrt{8 + \pi^2}}$$

15. (10 points) A particle is traveling in the x,y plane with velocity $\vec{v}(t) = \langle -2\sin(t), 2\cos(t) \rangle$ how fast is the particle moving? How far did the particle move from time $t = \pi/2$ to $t = \pi$?

$$\begin{aligned} \text{Speed} &= |\vec{v}(t)| = \sqrt{(-2\sin(t))^2 + (2\cos(t))^2} = \sqrt{4\sin^2(t) + 4\cos^2(t)} \\ &= \sqrt{4(\underbrace{\sin^2(t) + \cos^2(t)}_{=1})} = \sqrt{4} = 2 \end{aligned}$$

How far did the particle move from $t = \pi/2$ to $t = \pi$

$$\begin{aligned} \int_{\pi/2}^{\pi} \langle -2\sin(t), 2\cos(t) \rangle &= \left\langle \int_{\pi/2}^{\pi} -2\sin(t), \int_{\pi/2}^{\pi} 2\cos(t) \right\rangle \\ &= \langle 2\cos(t), 2\sin(t) \rangle \Big|_{\pi/2}^{\pi} \end{aligned}$$

$$= \langle 2\cos(\pi), 2\sin(\pi) \rangle - \langle 2\cos(\pi/2), 2\sin(\pi/2) \rangle$$

$$= \langle -2, 0 \rangle - \langle 0, 2 \rangle = \langle -2, -2 \rangle$$

$$|\langle -2, -2 \rangle| = \sqrt{2^2 + 2^2} = \sqrt{4 + 4} = \boxed{\sqrt{8}}$$

This is the displacement vector. Its magnitude will be the total distance traveled

16. (10 points) Find the following vectors using the vector function $\vec{r}(t) = \langle 2\sin(t), 5t, 2\cos(t) \rangle$

a. (5 pts) Find the unit tangent vector $\vec{T}(t)$

Recall

$$\vec{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

$$\vec{r}'(t) = \frac{d}{dt} \vec{r}(t) = \langle 2\cos(t), 5, -2\sin(t) \rangle$$

$$|\vec{r}'(t)| = \sqrt{(2\cos(t))^2 + 5^2 + (-2\sin(t))^2}$$

$$= \sqrt{4\cos^2(t) + 25 + 4\sin^2(t)}$$

$$= \sqrt{4(\underbrace{\cos^2(t) + \sin^2(t)}_{=1}) + 25} \Rightarrow \vec{T}(t) = \frac{1}{\sqrt{29}} \langle 2\cos(t), 5, -2\sin(t) \rangle$$

$$= \sqrt{4 + 25}$$

$$= \sqrt{29}$$

$$= \left\langle \frac{2\cos(t)}{\sqrt{29}}, \frac{5}{\sqrt{29}}, \frac{-2\sin(t)}{\sqrt{29}} \right\rangle$$

b. (5 pts) Find the unit Normal vector $\vec{N}(t)$

Recall

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$$

$$\vec{T}'(t) = \langle -2\sin(t), 0, -2\cos(t) \rangle$$

$$|\vec{T}'(t)| = \sqrt{(-2\sin(t))^2 + 0^2 + (-2\cos(t))^2}$$

$$= \sqrt{4\sin^2(t) + 4\cos^2(t)} = \sqrt{4(\underbrace{\sin^2(t) + \cos^2(t)}_{=1})}$$

$$= \sqrt{4} = 2$$

$$\Rightarrow \vec{N}(t) = \frac{1}{2} \langle -2\sin(t), 0, -2\cos(t) \rangle$$

$$\vec{N}(t) = \left\langle \frac{-2\sin(t)}{2}, \frac{0}{2}, \frac{-2\cos(t)}{2} \right\rangle = \langle -\sin(t), 0, -\cos(t) \rangle$$

c. (5 pts) Find the curvature κ

Recall

$$\kappa = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|^3} = \frac{|\vec{r}'(t) \times \vec{r}''(t)|}{|\vec{r}'(t)|^3}$$

↑
I'll use this one

$$|\vec{T}'(t)| = 2$$

$$|\vec{r}'(t)| = \sqrt{29}$$

$$\kappa = \frac{2}{\sqrt{29}}$$

17. (10 points) Consider the function $f(x, y) = x^3 - y$

a. (5 pts) Draw a contour map of the of $f(x, y)$

Choose z values to fix

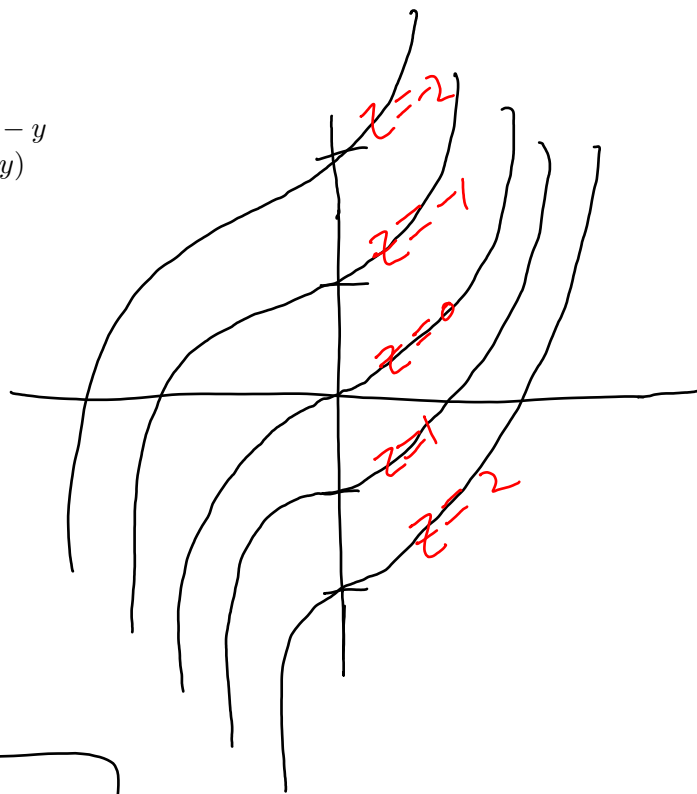
$$z = -2 \Rightarrow -2 = x^3 - y \rightarrow y = x^3 + 2$$

$$z = -1 \Rightarrow -1 = x^3 - y \rightarrow y = x^3 + 1$$

$$z = 0 \rightarrow 0 = x^3 - y \rightarrow y = x^3$$

$$z = 1 \rightarrow 1 = x^3 - y \rightarrow y = x^3 - 1$$

$$z = 2 \rightarrow 2 = x^3 - y \rightarrow y = x^3 - 2$$



b. (5 pts) Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} (x^3 - y) = 3x^2$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} (x^3 - y) = -1$$

c. (5 pts) Find $\frac{\partial^2 f}{\partial y \partial x}$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \frac{\partial}{\partial x} (x^3 - y)$$

$$= \frac{\partial}{\partial y} 3x^2 = \boxed{0}$$

Here we will use the chain rule $\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t}$

and $\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s}$

18. (10 points)

Suppose $f(x, y, z) = 3x^2 + 4xy^4 + 3xyz$ and $x = s \cos(3t)$, $y = sy^3$, $z = t \cos(s)$

a. (5 pts) Find $\frac{\partial f}{\partial t}$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial t} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial t} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial t}$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x} (3x^2 + 4xy^4 + 3xyz) = 6x + 4y^4 + 3yz$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y} (3x^2 + 4xy^4 + 3xyz) = 16xy^3 + 3xz$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z} (3x^2 + 4xy^4 + 3xyz) = 3xy$$

$$\frac{\partial x}{\partial t} = \frac{\partial}{\partial t} (s \cos(3t)) = -3s \sin(3t)$$

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial t} (sy^3) = 0$$

$$\frac{\partial z}{\partial t} = \frac{\partial}{\partial t} (t \cos(s)) = \cos(s)$$

b. (5 pts) Find $\frac{\partial f}{\partial s}$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \frac{\partial y}{\partial s} + \frac{\partial f}{\partial z} \frac{\partial z}{\partial s}$$

we have $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ above

$$\frac{\partial x}{\partial s} = \frac{\partial}{\partial s} (s \cos(3t)) = \cos(3t)$$

$$\frac{\partial y}{\partial s} = \frac{\partial}{\partial s} (sy^3) = y^3$$

$$\frac{\partial z}{\partial s} = \frac{\partial}{\partial s} (t \cos(s)) = -t \sin(s)$$

Putting it all together..

$$\frac{\partial f}{\partial t} = (6x + 4y^4 + 3yz)(-3s \sin(3t))$$

$$+ (16xy^3 + 3xz)(0)$$

$$+ (3xy) \cos(s)$$

$$\frac{\partial f}{\partial s} = (6x + 4y^4 + 3yz) \cos(3t) + (16xy^3 + 3xz)(y^3)$$

$$+ 3xy(-t \sin(s))$$

c. (5 pts) Find $\frac{\partial z}{\partial x}$

From the chain rule
we get the formula
for Implicit differentiation...

$$\frac{\partial z}{\partial x} = \frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}}$$

From above

$$\frac{\partial f}{\partial x} = 6x + 4y^4 + 3yz$$

$$\frac{\partial f}{\partial z} = 3xy$$

$$\frac{\partial z}{\partial x} = \frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}} = \frac{6x + 4y^4 + 3yz}{3xy}$$

19. (10 points)

Find the equation of the tangent plane to the surface $f(x, y) = x^2 + xy^3$ at the point $(1, 1, 2)$

we showed the equation of a tangent plane
at (x_0, y_0, z_0) is given by:

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$f_x = \frac{\partial}{\partial x} (x^2 + xy^3) = 2x + 3xy^2$$

$$f_y = \frac{\partial}{\partial y} (x^2 + xy^3) = 3xy^2$$

$$(x_0, y_0, z_0) = (1, 1, 2)$$

$$f_x(1, 1) = 2(1) + 3(1)(1) = 5$$

$$f_y(1, 1) = 3(1)(1) = 3$$

The plane is given by

$$z - 2 = 5(x - 1) + 3(y - 1)$$

20. (5 points) (Bonus all or nothing) Find a solution (by Guessing or looking one up before hand) to Laplace's equation $f_{xx} + f_{yy} = 0$ State your solution and show that it solves the PDE.

Google will know!

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Scrap Page

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