Practice Final

Math 1320 001 Summer 2014
C.Sampson
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Name and Unid: _____
by writing my name I swear by the honor code

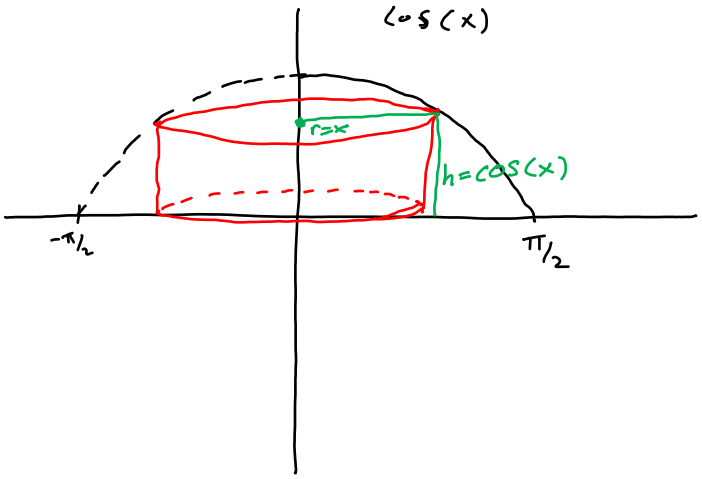
Read all of the following information before starting the exam:
Note this is the practice exam ignore point values on problems there are extras but the exam will look similar

- Show all work, clearly and in order, if you want to get full credit. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Justify your answers algebraically whenever possible to ensure full credit. Sketch all relevant graphs and explain all relevant mathematics.
- Circle or otherwise indicate your final answers.
- You MAY NOT use a calculator or notes. You may use the formulas provided.
- This test has 11 problems and one bonus problem and is worth 100 points, plus some extra credit at the end. It is your responsibility to make sure that you have all of the pages!
- Good luck!

Good Luck!
Here are some Formulas and rules for you.

This problem can be done with disks integrating along the y-axis but you then have to integrate $\arccos^2(y)$..... In this case the method shells is simpler.

1. (*10 points*)
Find the volume of the solid obtained by rotating the region bound by the curves $y = \cos(x)$, $y = 0$ when rotated around the y-axis.



$SA = 2\pi r h$

$r = x \quad h = \cos(x)$

$V = \int_0^{\pi/2} 2\pi r h = \int_0^{\pi/2} 2\pi x\cos(x)$

$\int_0^{\pi/2} 2\pi x\cos(x)dx = 2\pi\int_0^{\pi/2} x\cos(x)\,dx$

we will use integration by parts

$u = x \quad dv = \cos(x)$

$du = dx \quad v = \sin(x)$

$2\pi\int_0^{\pi/2} x\cos(x)dx = 2\pi\left(x\sin(x)\Big|_0^{\pi/2} - \int_0^{\pi/2}\sin(x)\,dx\right)$

$= 2\pi\left[x\sin(x)\Big|_0^{\pi/2} - (-\cos(x))\Big|_0^{\pi/2}\right]$

$= 2\pi\left[x\sin(x)\Big|_0^{\pi/2} + \cos(x)\Big|_0^{\pi/2}\right]$

$= 2\pi\left[\pi/2\,\sin(\pi/2) - 0(\sin(0)) + \cos(\pi/2) - \cos(0)\right]$ (sin(π/2) = 1, sin(0) = 0, cos(π/2) = 0, cos(0) = 1)

$= 2\pi\left[\pi/2 - 1\right] = 2\pi\,\pi/2 - 2\pi$

$= \boxed{\pi^2 - 2\pi}$

2. (*10 points*) The force on a particle moving along the x-axis from $x = 0$ to $x = 1$ is given by $F(x) = x\sqrt{x^2+1}$ find the work done on the particle.

Recall $W = \int_a^b F(x)\,dx$

we can do this with a u substitution.

$W = \int_0^1 x\sqrt{x^2+1}\,dx$

$u = x^2+1$

$\frac{du}{dx} = 2x \longrightarrow dx = \frac{du}{2x}$

$W = \int \cancel{x}\sqrt{u}\,\frac{du}{2\cancel{x}} = \int \sqrt{u}\,du = \int u^{1/2}\,du$

$= \frac{u^{1/2}}{1/2} = 2\sqrt{u} = 2\sqrt{x^2+1}$

So $\int_0^1 x\sqrt{x^2+1}\,dx = 2\sqrt{x^2+1}\Big|_0^1 = 2\sqrt{1^2+1} - 2\sqrt{0^2+1}$

$= 2\sqrt{2} - 2\sqrt{1} = \boxed{2\sqrt{2}-2}$ Newtons

3. (*10 points*) Verify that $y = cos(kx)$ is a solution to the differential equation $y'' + k^2y = 0$ Note don't try to solve just verify.

To verify we simply plug $y=\cos(kx)$ into the DE

$y'' + k^2 y = 0$

$y = \cos(kx)$

$y' = -k\sin(kx)$

$y'' = -k^2\cos(kx)$

$-k^2\cos(kx) + k^2\cos(kx) = 0$ ✓

4. (*15 points*)

a. (*5 pts*) Solve the initial value problem for the differential equation $\frac{dy}{dx} = x^2$ with y(0)=1

Here we can simply integrate both sides and use $y(0)=1$ to find the constant of integration

$\frac{dy}{dx} = x^2$

$\int \frac{dy}{dx}\,dx = \int x^2\,dx$

$y = \frac{x^3}{3} + C$

then $y(0) = 1$

thus when $x=0$ y must $= 1$

$1 = \frac{0^3}{3} + C$

$\Rightarrow 1 = C$

Thus:

$$y = \frac{x^3}{3} + 1$$

b. (*10 pts*) Find the general solution to the differential equation $\cos(x)h' = \sin(x)$

Here it is implied that h is a function of x. First we must seperate the variables Get all of the "h" stuff on one side and the "x" stuff on the other. Then we can integrate both sides. The term general solution means we won't solve for the constant of integration.

$\cos(x)\,h' = \sin(x)$

$h' = \frac{\sin(x)}{\cos(x)}$ now we can integrate both sides

$\int h'\,dx = \int \frac{\sin(x)}{\cos(x)}\,dx$

$u = \cos(x)$

$\frac{du}{dx} = -\sin(x)$

$\Rightarrow dx = \frac{du}{-\sin(x)}$

$\Rightarrow h = \int \frac{\sin(x)}{u}\frac{du}{-\sin(x)}$

$\Rightarrow h = \int \frac{-1}{u}\,du = -\ln(|u|) + C$

$\Rightarrow h = -\ln(|u|) + C$

$\Rightarrow$ $$h = -\ln(|\cos(x)|) + C$$

5. (*20 points*) A population of bacteria grows exponentially and starts with 2 cells. After 10 minuets there are 8 cells.

a. (*5 pts*) Write down the equation that gives the population as a function of time

Exponential growth can be modeled by:

$P = P_0 e^{kt}$

P_0: Initial Population; k: growth rate

we can start with this and use the given information to find P_0, k

From the problem $P_0 = 2$ to find k we use the fact that $P = 8$ after 10 min

$P = 2e^{kt} \rightarrow P(10) = 8 = 2e^{k(10)}$

$8 = 2e^{k(10)}$

$4 = e^{k(10)}$

$\ln(4) = \ln\left(e^{k(10)}\right)$

$\ln(4) = k(10) \Rightarrow k = \frac{\ln(4)}{10}$

$\Rightarrow \boxed{P = 2e^{\frac{\ln(4)}{10}t}}$

If you like... $= 2\left(e^{\ln(4)}\right)^{\frac{1}{10}t} = 2(4)^{1/10\,t}$

b. (*5 pts*) When will the population reach 75?

Here we set $P = 75$ then solve for t.

$75 = 2e^{\frac{\ln(4)}{10}t}$

$\Rightarrow \frac{75}{2} = e^{\frac{\ln(4)}{10}t}$

$\Rightarrow \ln\left(\frac{75}{2}\right) = \ln\left(e^{\frac{\ln(4)}{10}t}\right)$

$\Rightarrow \ln\left(\frac{75}{2}\right) = \frac{\ln(4)}{10}t$

$\Rightarrow \boxed{t = \frac{10\ln\left(\frac{75}{2}\right)}{\ln(4)}}$

6. (*10 points*) Determine wether the given series is convergent or divergent. If it is convergent find it's sum

a. (*5 pts*) $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n + \frac{1}{n} - \frac{1}{n+1}$

Here we need to split the sum to analyize it.

$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n + \frac{1}{n} - \frac{1}{n+1}$

$= \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n + \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$

↓ Geometric ↓ Telescoping

Since $r = \frac{1}{3} < 1$ we can use $\sum_{n=1}^{\infty} a_1 r^n = \frac{a_1 r}{1-r}$

For this sum we can write out the first few terms to see it converges.

$$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n + \frac{1}{n} - \frac{1}{n+1} = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n + \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$$

$$= \frac{1/3}{1-1/3} + \left(\frac{1}{1}-\frac{1}{2}\right) + \left(\frac{1}{2}-\frac{1}{3}\right) + \left(\frac{1}{3}-\frac{1}{4}\right) + \left(\frac{1}{4}-\frac{1}{5}\right) + \cdots$$

$$= \frac{1/3}{2/3} + 1 - \cancel{1/2} + \cancel{1/2} - \cancel{1/3} + \cancel{1/3} - \cancel{1/4} + \cancel{1/4} - \cancel{1/5} + \cancel{1/5} + \cdots$$

$$= \frac{1}{3}\cdot\frac{3}{2} + 1 = \frac{1}{2} + 1 = \boxed{\frac{3}{2}}$$

b. (*5 pts*) $\sum_{n=1}^{\infty} \frac{n^2+3n^6}{n^3+n^6}$

Let's use the limit test check for diversence...

recall $\sum_{n=1}^{\infty} a_n$ diverges if $\lim_{n\to\infty} a_n \neq 0$

$$\lim_{n\to\infty} \frac{n^2+3n^6}{n^3+n^6} = \frac{\infty}{\infty}$$

You can use L'Hopital's rule if you like but i'll do something different.

$$= \lim_{n\to\infty} \frac{\frac{1}{n^6}(n^2+3n^6)}{\frac{1}{n^6}(n^3+n^6)} = \lim_{n\to\infty} \frac{\frac{n^2}{n^6}+\frac{3n^6}{n^6}}{\frac{n^3}{n^6}+\frac{n^6}{n^6}}$$

$$= \lim_{n\to\infty} \frac{\frac{1}{n^4}+3}{\frac{1}{n^3}+1} = \frac{\lim_{n\to\infty}\frac{1}{n^4} + \lim_{n\to\infty} 3}{\lim_{n\to\infty}\frac{1}{n^3} + \lim_{n\to\infty} 1} = \frac{0+3}{0+1} = 3 \neq 0$$

($\lim \frac{1}{n^4} = 0$, $\lim 3 = 3$, $\lim \frac{1}{n^3} = 0$, $\lim 1 = 1$)

$\Rightarrow \sum_{n=1}^{\infty} \frac{n^2+3n^6}{n^3+n^6}$ diverges

c. (*5 pts*) $\sum_{n=0}^{\infty} \frac{2^n}{n!}$

To show this series converges we can use the ratio test

Recall If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$ the series converges.

Also we can recall that

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ If we plug in 2 for x then

$e^2 = \sum_{n=0}^{\infty} \frac{2^n}{n!}$ which converges

Since the taylor series for e^x converges for all x.

Ratio test: $a_n = \frac{2^n}{n!}$ $a_{n+1} = \frac{2^{n+1}}{(n+1)!}$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{2^{n+1}}{(n+1)!}}{\frac{2^n}{n!}}\right|$$

$$= \lim_{n\to\infty} \left|\frac{\cancel{2^{n+1}}}{(n+1)\cancel{n!}} \cdot \frac{\cancel{n!}}{\cancel{2^n}}\right| = \lim_{n\to\infty} \left|\frac{2}{n+1}\right| = 0 < 1$$

Thus $\sum_{n=0}^{\infty} \frac{2^n}{n!}$ converges by the ratio test

Further $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ $\Rightarrow \boxed{\sum_{n=1}^{\infty} \frac{2^n}{n!} = e^2}$

7. (*10 points*) Determine wether or not the given series is absolutely convergent , conditionally convergent , or divergent. $\sum_{n=1}^{\infty}(-1)^n \frac{1}{3n}$

Recall that absolut convergence means that the series converges without the alternating part where as conditional convergence means the series diverges unless the terms are alternating.

Check for convergence in absolute value

$\sum_{n=1}^{\infty}\left|(-1)^n\frac{1}{3n}\right| = \sum_{n=1}^{\infty}\frac{1}{3n} = \frac{1}{3}\sum_{n=1}^{\infty}\frac{1}{n}$ ← harmonic series

which diverges.

Check for convergence with Alternating form

I'll use the Alternating series test

$\lim_{n\to\infty}\left|(-1)^n\frac{1}{3n}\right| = \lim_{n\to\infty}\frac{1}{3n} = 0$ Thus the alternating series converges. $\Rightarrow$ conditionally convergent

8. (*10 points*) Find the radius of convergence and interval of convergence for the following series.

a. (*5 pts*) $\sum_{n=1}^{\infty}(x-2)^n$

For radius of convergence problems the ratio test helps. we can apply the ratio test then choose our x values to ensure that the limit is less than 1 to give convergence.

Ratio test: we want $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| < 1$

$a_n = (x-2)^n \quad a_{n+1} = (x-2)^{n+1}$

for convergence we want $|x-2|<1$

$\Rightarrow \lim_{n\to\infty}\left|\frac{(x-2)^{n+1}}{(x-2)^n}\right| = \lim_{n\to\infty}|(x-2)| = |x-2| < 1 \Rightarrow R = 1$

checking Endpoints: $|x-2|<1 \Rightarrow -1 < x-2 < 1 \Rightarrow -1 < x < 3$

check $x=-1$: $\sum_{n=1}^{\infty}(-1-2)^n = \sum_{n=1}^{\infty}(-3)^n = \sum_{n=1}^{\infty}(-1)^n 3^n$; $\lim_{n\to\infty} 3^n = \infty \neq 0 \Rightarrow$ diverges

check $x=3$: $\sum_{n=1}^{\infty}(3-2)^n = \sum_{n=1}^{\infty}(1)^n$; $\lim_{n\to\infty}(1)^n = 1 \neq 0 \Rightarrow$ diverges

Interval of convergence is $(-1, 3)$

b. (*5 pts*) $\sum_{n=1}^{\infty}\frac{(-1)^n x^n}{n!}$

Ratio test

$a_n = \frac{(-1)^n x^n}{n!} \quad a_{n+1} = \frac{(-1)^{n+1} x^{n+1}}{(n+1)!}$

For this problem we get 0 for our ratio test limit regardless of the x-value since $0<1$ This will converge for any x value

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}x^{n+1}}{(n+1)!}\cdot\frac{n!}{(-1)^n x^n}\right| = \lim_{n\to\infty}\left|\frac{(-1)x}{n+1}\right| = 0 < 1$ for all x

$\Rightarrow R = \infty$ Interval of convergence $(-\infty, \infty)$

c. (*5 pts*) $\sum_{n=1}^{\infty}\frac{(-2)^n(x-2)^n}{n}$

Ratio test again.

$a_n = \frac{(-2)^n(x-2)^n}{n} \quad a_{n+1} = \frac{(-2)^{n+1}(x-2)^{n+1}}{n+1}$

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{\frac{(-2)^{n+1}(x-2)^{n+1}}{n+1}}{\frac{-2^n(x-2)^n}{n}}\right| = \lim_{n\to\infty}\left|\frac{(-2)^{n+1}(x-2)^{n+1}}{n+1}\cdot\frac{n}{(-2)^n(x-2)^n}\right| = \lim_{n\to\infty}\left|\frac{(-2)^{n+1}(x-2)^{n+1}(n+1)}{(-2)^n(x-2)^n\, n}\right|$

$= \lim_{n\to\infty}\left|\frac{(-2)(x-2)\,n}{n+1}\right| = |-2(x-2)|\lim_{n\to\infty}\left|\frac{n}{n+1}\right| = |-2(x-2)| = |2(x-2)| = 2|x-2| < 1 \Rightarrow |x-2| < \frac{1}{2}$ ($\lim_{n\to\infty}\frac{n}{n+1} = 1$)

$R = \frac{1}{2}$

Thn $|x-2| < \frac{1}{2} \Rightarrow -\frac{1}{2} < x-2 < \frac{1}{2} \Rightarrow 2-\frac{1}{2} < x < \frac{1}{2}+2 \Rightarrow \frac{3}{2} < x < \frac{5}{2}$

check $x = \frac{3}{2}$

$\sum_{n=1}^{\infty}\frac{(-2)^n(\frac{3}{2}-2)^n}{n} = \sum_{n=1}^{\infty}\frac{(-2)^n(-\frac{1}{2})^n}{n}$

$= \sum_{n=1}^{\infty}\frac{(-2(-\frac{1}{2}))^n}{n} = \sum_{n=1}^{\infty}\frac{1^n}{n} = \sum_{n=1}^{\infty}\frac{1}{n}$

which is the harmonic series and thus diverges.

9. (*10 points*) Find a power series for the following function using the geometric series. $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ when $|x| < 1$

a. (*5 pts*) $f(x) = \arctan(x)$

For this problem we want to manipulate $\frac{1}{1-x}$ to get $\arctan(x)$

what ever we do to $\frac{1}{1-x}$ we also do to the series as they are equal.

In this case we know $\int \frac{1}{1+x^2}\,dx = \arctan(x)$

So we will plug in $-x^2$ then integrate.

$$\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$$

$$\Rightarrow \sum_{n=0}^{\infty} (-x^2)^n = \frac{1}{1-(-x^2)}$$

$$\Rightarrow \sum_{n=0}^{\infty} (-1)^n x^{2n} = \frac{1}{1+x^2}$$

$$\Rightarrow \int \sum_{n=0}^{\infty} (-1)^n x^{2n}\,dx = \int \frac{1}{1+x^2}\,dx$$

$$\Rightarrow \sum_{n=0}^{\infty} \int (-1)^n x^{2n}\,dx = \int \frac{1}{1+x^2}\,dx$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = \arctan(x)$$

$$\arctan(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1}$$

10. (*10 points*)

Find taylor series out to 4 terms for the following functions at the given centers

a. (*5 pts*) $f(x) = e^x$ centered at $x = 1$

Here we will use taylor's theorem $g(x) = \sum_{n=0}^{\infty} \frac{g^{(n)}(a)}{n!}(x-a)^n$ where a is our center.

$f(x) = e^x$	$f(1) = e$
$f'(x) = e^x$	$f'(1) = e$
$f''(x) = e^x$	$f''(1) = e$
$f'''(x) = e^x$	$f'''(1) = e$

$$e^x \approx \frac{f^{(0)}(1)}{0!}(x-1)^0 + \frac{f'(1)}{1!}(x-1)^1 + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 + \cdots$$

$$\Rightarrow e^x \approx e + e(x-1) + \frac{e}{2}(x-1)^2 + \frac{e}{6}(x-1)^3 + \cdots$$

b. (*5 pts*) $f(x) = \sin(x)$ centered at $x = \pi$

$f(x) = \sin(x)$	$f(\pi) = 0$
$f'(x) = \cos(x)$	$f'(\pi) = -1$
$f''(x) = -\sin(x)$	$f''(\pi) = 0$
$f'''(x) = -\cos(x)$	$f'''(\pi) = 1$
$f^{(4)}(x) = \sin(x)$	$f^{(4)}(\pi) = 0$
$f^{(5)}(x) = \cos(x)$	$f^{(5)}(\pi) = -1$
$f^{(6)}(x) = -\sin(x)$	$f^{(6)}(\pi) = 0$
$f^{(7)}(x) = -\cos(x)$	$f^{(7)}(\pi) = 1$

$$\sin(x) \approx \frac{f(\pi)}{0!}(x-\pi)^0 + \frac{f'(\pi)}{1!}(x-\pi)^1 + \frac{f''(\pi)}{2!}(x-\pi)^2 + \frac{f'''(\pi)}{3!}(x-\pi)^3 + \frac{f^{(4)}(\pi)}{4!}(x-\pi)^4 + \frac{f^{5}(\pi)}{5!}(x-\pi)^5 + \frac{f^{6}(\pi)}{6!}(x-\pi)^6 + \frac{f^{7}(\pi)}{7!}(x-\pi)^7$$

$$\Rightarrow \sin(x) \approx -(x-\pi) + \frac{1}{3!}(x-\pi)^3 - \frac{1}{5!}(x-\pi)^5 + \frac{1}{7!}(x-\pi)^7$$

11. (*10 points*)
Calculate the indefinite integral $\int \frac{\tan^{-1}(x)}{x} dx$ using power series.

Here what we want to do is replace $\tan^{-1}(x)$ with it's powerseries and integrate.

The series for $\tan^{-1}(x)$ is

$$\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = \tan^{-1}(x)$$

$$\int \frac{\tan^{-1}(x)}{x} dx = \int \frac{1}{x} \tan^{-1}(x)\, dx = \int \frac{1}{x} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}\, dx$$

now we can multiply the $\frac{1}{x}$ into the series and simplify.

$$= \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{x(2n+1)} = \int \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n+1} dx = \sum_{n=0}^{\infty} \int (-1)^n \frac{x^{2n}}{2n+1} dx$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \int x^{2n} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \frac{x^{2n+1}}{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)^2}$$

12. (*10 points*)
Find the distance between the points $(1, 0, 6)$ and $(3, -2, 5)$

we can simply apply the distance formula $d = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

$$d = \sqrt{(3-1)^2+(-2-0)^2+(5-6)^2} = \sqrt{1^2+2^2+1^2}$$

$$= \boxed{\sqrt{5}}$$

13. (*10 points*)
Show that the equation represents a sphere, find it's center and radius. $x^2+y^2+z^2+7x-8y-6z = 0$

Here again we just want to complete the square in (x, y, z)

$$x^2+y^2+z^2+7x-8y-6z = 0$$

$$x^2+7x+y^2-8y+z^2-6z = 0$$

$$\left(x^2+7x+\frac{49}{4}-\frac{49}{4}\right)+\left(y^2-8y+16-16\right)+\left(z^2-6z+9-9\right) = 0$$

$$\left(x+\frac{7}{2}\right)^2-\frac{49}{4}+(y-4)^2-16+(z-3)^2-9 = 0$$

$$\left(x+\tfrac{7}{2}\right)^2+(y-4)^2+(z-3)^2 = 49/4+16+9 = \frac{49}{4}+\frac{64}{4}+\frac{36}{4} = \frac{149}{4}$$

$$\Rightarrow \left(x+\frac{7}{2}\right)^2+(y-4)^2+(z-3)^2 = \frac{149}{4}$$

center $\left(-\frac{7}{2}, 4, 3\right)$ radius: $\sqrt{\frac{149}{4}}$

14. (*10 points*) Find $\vec{a}+\vec{b}, \vec{a}-\vec{b}, |\vec{a}|, |\vec{a}-\vec{b}|$ for the vectors $\vec{a} =< 1,2,3 >, \vec{b} =< 4,-2,-1 >$

$\vec{a}+\vec{b} = \langle 1,2,3\rangle + \langle 4,-2,-1\rangle = \langle 5,0,2\rangle$

$\vec{a}-\vec{b} = \langle 1,2,3\rangle - \langle 4,-2,-1\rangle = \langle -3,4,4\rangle$

$|\vec{a}| = \sqrt{1^2+2^2+3^2} = \sqrt{1+4+9} = \sqrt{14}$

$|\vec{a}-\vec{b}| = \sqrt{(-3)^2+4^2+4^2} = \sqrt{9+16+16} = \sqrt{41}$

15. (*5 points*) Find the cross product between $\vec{a} =< 1, -2, 0 >$ and $\vec{b} =< 0, 1, 3 >$ and verify it is orthogonal to both $\vec{a}$ and $\vec{b}$

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ 1 & -2 & 0 \\ 0 & 1 & 3 \end{vmatrix} = i \begin{vmatrix} -2 & 0 \\ 1 & 3 \end{vmatrix} - j \begin{vmatrix} 1 & 0 \\ 0 & 3 \end{vmatrix} + k \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix}$$

$$= i\,(-2(3) - 1(0)) - j\,(1(3) - 0(0)) + k\,(1(1) - 0(-2))$$

$$= i\,(-6) - j\,(3) + k\,(1) = -6i - 3j + k$$

$$= \boxed{\langle -6, -3, 1 \rangle}$$

To show orthogonality with $\vec{a}, \vec{b}$ we use the dot product.

$$\langle 1, -2, 0 \rangle \cdot \langle -6, -3, 1 \rangle = -6 + 6 + 0 = 0 \quad \checkmark$$

$$\langle 0, 1, 3 \rangle \cdot \langle -6, -3, 1 \rangle = 0 + (-3) + 3 = 0 \quad \checkmark$$

16. (*15 points*)

a. (*5 pts*) Find the symmetric, parametric , and vector equations of the line between the points $(1, -2, 3)$ and $(2, 4, 2)$

To find $\vec{v}$ we can take the difference of the two points on the line.

$\vec{v} = (2,4,2) - (1,-2,3) = \langle 2-1, 4-(-2), 2-3 \rangle$

$\Rightarrow \vec{v} = \langle 1, 6, -1 \rangle = \langle a, b, c \rangle$

we can then choose either point to be (x_0, y_0, z_0) I'll pick $(1, -2, 3)$

Symmetric

$$\frac{x-x_0}{a} = \frac{y-y_0}{b} = \frac{z-z_0}{c}$$

Parametric

$x = x_0 + at \quad y = y_0 + bt \quad z = z_0 + ct$

Vector

$r(t) = \langle x_0 + at, y_0 + bt, z_0 + ct \rangle$

where (x_0, y_0, z_0) are a point on the line and $\vec{v} = \langle a, b, c \rangle$ is a vector parallel to the line.

Symmetric

$$\frac{x-1}{1} = \frac{y+2}{6} = \frac{z-3}{-1}$$

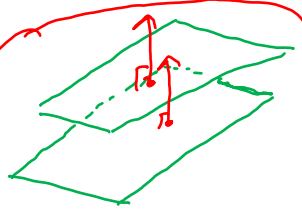
Parametric

$x = 1 + t \quad y = -2 + 6t \quad z = 3 - t$

Vector

$\vec{r}(t) = \langle 1+t, -2+6t, 3-t \rangle$

b. (*10 pts*) Find the equation of the plane parallel to the plane $2x + 3y - 4z = 2$ going through the point $(1, 2, 3)$



recall that parallel planes have normal vectors in the same direction and that one form for the equation of a plane is given by: $ax + by + cz = d$ where $\langle a, b, c \rangle = n$

Thus we can use the normal vector for the given plane to define the parallel plane.

$2x + 3y - 4z = 2$

$\downarrow \quad \downarrow \quad \downarrow$

$a \quad b \quad c$

$\vec{n} = \langle 2, 3, -4 \rangle$

$(x_0, y_0, z_0) = (1, 2, 3)$

so the plane is given by:

$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$

$\Rightarrow \boxed{2(x-1) + 3(y-2) - 4(z-3) = 0}$

17. (*10 points*) Find the distance from the point $(4, 1, -2)$ to the plane $2x + 3y + 4z = 1$

Recall the distance from a point (x_1, y_1, z_1) to a plane $ax+by+cz=d$ is given by:
$D = \dfrac{|ax_1+by_1+cz_1-d|}{\sqrt{a^2+b^2+c^2}}$

$$D = \frac{|2(4)+3(1)+4(-2)-1|}{\sqrt{2^2+3^2+4^2}}$$

$$= \frac{|8+3-8-1|}{\sqrt{4+9+16}} = \frac{|2|}{\sqrt{29}} = \boxed{\frac{2}{\sqrt{29}}}$$

18. (*15 points*) Do the following coordinate changes and plots

a. (*5 pts*) Change $(1, \sqrt{(2)}/2, \sqrt{(2)}/2)$ from Rectangular to spherical coordinates.

Recall
$\rho = \sqrt{x^2+y^2+z^2}$
$\tan\phi = \frac{\sqrt{x^2+y^2}}{z}$
$\tan\theta = \frac{y}{x}$

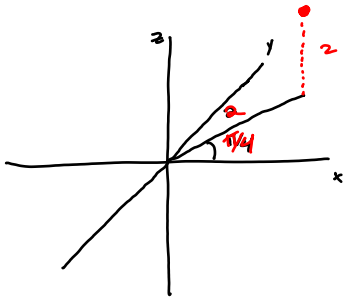
$$\rho = \sqrt{1^2+\left(\frac{\sqrt{2}}{2}\right)^2+\left(\frac{\sqrt{2}}{2}\right)^2} = \sqrt{1+\frac{2}{4}+\frac{2}{4}} = \sqrt{1+1} = \boxed{\sqrt{2}}$$

$$\tan\phi = \frac{\sqrt{x^2+y^2}}{z} = \frac{\sqrt{1+\left(\frac{\sqrt{2}}{2}\right)^2}}{\frac{\sqrt{2}}{2}} = \frac{\sqrt{1+\frac{2}{4}}}{\frac{\sqrt{2}}{2}} = \frac{\sqrt{\frac{3}{2}}}{\frac{\sqrt{2}}{2}} = \frac{\sqrt{3}}{\sqrt{2}}\cdot\frac{2}{\sqrt{2}} = \frac{\sqrt{3}(2)}{2} = \sqrt{3} \Rightarrow \tan\phi = \sqrt{3}$$
$$\Rightarrow \boxed{\phi = \pi/6}$$

$$\tan\theta = \frac{y}{x} = \frac{\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = 1 \Rightarrow \tan\theta = 1 \Rightarrow \boxed{\theta = \pi/4}$$

$$\boxed{\left(1, \frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \rightarrow \left(\sqrt{2}, \pi/4, \pi/6\right)}$$

b. (*5 pts*) Plot the point given in cylindrical coordinates as $(2, \frac{\pi}{4}, 2)$ Then convert it to rectangular coordinates.



$\uparrow r \uparrow \theta \uparrow z$

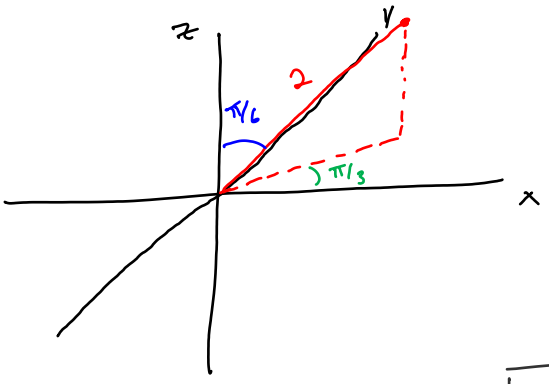
Recall
$x = r\cos\theta$
$y = r\sin\theta$
$z = z$

$$x = 2\cos\pi/4 = 2\sqrt{2}/2 = \sqrt{2}$$
$$y = 2\sin\pi/4 = 2\sqrt{2}/2 = \sqrt{2}$$
$$z = 2$$

$$\boxed{(2, \pi/4, 2) \rightarrow (\sqrt{2}, \sqrt{2}, 2)}$$

c. (*5 pts*) Plot the point in spherical coordinates given by $(2, \frac{\pi}{3}, \frac{\pi}{6})$ Then convert it to rectangular coordinates.

$\uparrow \rho \uparrow \theta \uparrow \phi$



Recall
$x = \rho\sin\phi\cos\theta$
$y = \rho\sin\phi\sin\theta$
$z = \rho\cos\phi$

$$x = 2\sin\pi/6\cos\pi/3 = 2 \cdot 1/2 \cdot 1/2 = 1/2$$
$$y = 2\sin\pi/6\sin\pi/3 = 2 \cdot 1/2 \cdot \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}$$
$$z = 2\cos\pi/6 = 2\frac{\sqrt{3}}{2} = \sqrt{3}$$

$$\boxed{(2, \pi/3, \pi/6) \rightarrow \left(1/2, \frac{\sqrt{3}}{2}, \sqrt{3}\right)}$$

19. (*10 points*) Find the velocity ,acceleration and speed of a particle whose position is given by $\vec{r}(t) =< 1+t, t^2, cos(\pi t) >$ at $t = 1$

recall:

$\vec{v}(t) = \frac{d}{dt}\vec{r}(t)$

$\vec{a}(t) = \frac{d}{dt}\vec{v}(t)$

also that the speed $v = |\vec{v}(t)|$

$$\vec{v}(t) = \frac{d}{dt}\langle 1+t, t^2, \cos \pi t\rangle$$
$$= \left\langle \frac{d}{dt}(1+t), \frac{d}{dt}t^2, \frac{d}{dt}\cos \pi t \right\rangle$$
$$= \langle 1, 2t, -\pi \sin \pi t\rangle$$

$$\vec{a}(t) = \frac{d}{dt}\vec{v}(t) = \frac{d}{dt}\langle 1, 2t, -\pi \sin \pi t\rangle$$
$$= \langle 0, 2, -\pi^2 \cos \pi t\rangle$$

at $t = 1$

$$\vec{v}(1) = \langle 1, 2(1), -\pi \sin(\pi(1))\rangle = \langle 1, 2, 0\rangle \quad (\sin(\pi) = 0)$$
$$\vec{a}(1) = \langle 0, 2, -\pi^2 \cos(\pi(1))\rangle = \langle 0, 2, \pi^2\rangle \quad (\cos(\pi) = -1)$$

$$\text{speed} = |\vec{v}(1)| = \sqrt{1^2 + 2^2 + 0^2} = \sqrt{5}$$

20. (*10 points*) A particle is traveling in the x,y plane with velocity $\vec{v}(t) =< -2, 2 >$ how fast is the particle moving? How far did the particle move from time $t = 1$ to $t = 3$?

Recall that speed $= |\vec{v}(t)|$ we can get the distance traveled by calculating the arclength. recall that arclength for a position vector function $\vec{r}(t) = \langle x(t), y(t)\rangle$

$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$

here we are given velocity which is the derivitive of position. so we have

$\frac{dx}{dt} = -2$, $\frac{dy}{dt} = 2$

$$L = \int_1^3 \sqrt{(-2)^2 + (2)^2}\, dt$$
$$= \int_1^3 \sqrt{4+4}\, dt = \int_1^3 \sqrt{8}\, dt$$
$$= \sqrt{8}t\Big|_1^3 = \sqrt{8}(3) - \sqrt{8}$$
$$= 3\sqrt{8} - \sqrt{8} = 2\sqrt{8}$$

21. (*10 points*) Find the following vectors using the vector function $\vec{r}(t) = < sin(t), 1, cos(t) >$

a. (*5 pts*) Find the unit tangent vector $\vec{T}(t)$

Recall that $\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$

$\vec{r}'(t) = \frac{d}{dt}\vec{r}(t) = \frac{d}{dt} < \sin(t), 1, \cos(t) > = < \cos(t), 0, -\sin(t) >$

$|\vec{r}'(t)| = \sqrt{\cos^2(t) + 0^2 + (-\sin(t))^2} = \sqrt{\cos^2 t + \sin^2 t} = \sqrt{1} = 1$

$\Rightarrow \boxed{\vec{T}(t) = \frac{< \cos(t), 0, -\sin(t) >}{1} = < \cos(t), 0, -\sin(t) >}$

b. (*5 pts*) Find the unit Normal vector $\vec{N}(t)$

Recall that $\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$

$\vec{T}'(t) = \frac{d}{dt}\vec{T}'(t) = \frac{d}{dt} < \cos(t), 0, -\sin(t) > = < -\sin t, 0, -\cos(t) >$

$|\vec{T}'(t)| = \sqrt{\sin^2 t + 0^2 + \cos^2(t)} = 1$

$\Rightarrow \boxed{\vec{N}(t) = \frac{< -\sin(t), 0, -\cos(t) >}{1} = < -\sin(t), 0, -\cos(t) >}$

c. (*5 pts*) Find the curvature κ

Recall $\kappa = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$

$\kappa = \frac{|T'(t)|}{|r'(t)|} = \frac{1}{1} = \boxed{1}$

Here we will use the chain rule for functions of 3 variables each of which depends on two parameters.

22. (*10 points*)
Suppose $f(x,y,z) = 2xy^4 + 3xyz$ and $x = st^2$, $y = s^2t$, $z = t$

a. (*5 pts*) Find $\frac{\partial f}{\partial t}$

$$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(2xy^4+3xyz) = 2y^4+3yz$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(2xy^4+3xyz) = 8xy^3+3xz$$

$$\frac{\partial f}{\partial z} = \frac{\partial}{\partial z}(2xy^4+3xyz) = 3xy$$

$$\frac{\partial x}{\partial t} = \frac{\partial}{\partial t}(st^2) = 2st \qquad \frac{\partial z}{\partial t} = 1$$

$$\frac{\partial y}{\partial t} = \frac{\partial}{\partial t}(s^2t) = s^2$$

$$\Rightarrow \boxed{\frac{\partial f}{\partial t} = (2y^4+3yz)2st + (8xy^3+3xz)s^2 + 3xy(1)}$$

b. (*5 pts*) Find $\frac{\partial f}{\partial s}$

$$\frac{\partial f}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial s}$$

we have $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ above

$$\frac{\partial x}{\partial s} = t^2 \qquad \frac{\partial y}{\partial s} = 2st \qquad \frac{\partial z}{\partial s} = 0$$

$$\Rightarrow \boxed{\frac{\partial f}{\partial s} = (2y^4+3yz)t^2 + (8xy^3+3xz)(2st) + 3xy(0)}$$

c. (*5 pts*) Find $\frac{\partial z}{\partial x}$ ← Here we are assuming z can be written in terms of x.

Recall $\frac{\partial z}{\partial x} = \frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}}$

Think of them like fractions...

$$\frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}} = \frac{\partial f}{\partial x}\cdot\frac{\partial z}{\partial f} = \frac{\partial z}{\partial x}$$

$$\frac{\partial f}{\partial x} = 2y^4+3yz$$

$$\frac{\partial f}{\partial z} = 3xy$$

$$\Rightarrow \boxed{\frac{\partial z}{\partial x} = \frac{\frac{\partial f}{\partial x}}{\frac{\partial f}{\partial z}} = \frac{2y^4+3yz}{3xy}}$$

23. (*10 points*)
Find the equation of the tangent plane to the surface $f(x,y) = x^2y + y^2x + 2x$ at the point $(1,1,2)$

Recall that the equation of a tangent plane at (x_0, y_0, z_0) for a function of two variables is given by:

$$z - z_0 = f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2y + y^2x + 2x)$$

$$= 2xy + y^2 + 2$$

$$f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2y + y^2x + 2x)$$

$$= x^2 + 2yx$$

$$\Rightarrow f_x|_{(1,1)} = 2(1)(1) + 1^2 + 2 = 5$$

$$f_y|_{(1,1)} = 1^2 + 2(1)(1) = 3$$

$$x_0 = 1 \quad y_0 = 1 \quad z_0 = 2$$

$$\Rightarrow \boxed{z - 2 = 5(x-1) + 3(y-1)}$$

24. (*10 points*) Find the the relative maximums, minimums, or saddle points for the function $f(x,y) = x^2 - 2xy + 3y^2$

for this problem recall that we need to find where $\nabla f = \langle 0, 0 \rangle$ that is the point (x_0, y_0) where $f_x = 0$ and $f_y = 0$. Then we can use the second partials test to determine what kind of points we have.

Second partials test:

$D = f_{xx}f_{yy} - (f_{xy})^2$

If $D > 0$ and $f_{xx} > 0 \Rightarrow$ min

If $D > 0$ and $f_{xx} < 0 \Rightarrow$ max

If $D < 0 \Rightarrow$ saddle pt.

If $D = 0$ inconclusive

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y} \right\rangle$$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}(x^2 - 2xy + y^2) = 2x - 2y$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}(x^2 - 2xy + y^2) = -2x + 6y$$

So we want

$$2x - 2y = 0 \qquad -2x + 6y = 0$$

We can treat this as a linear system of equations:

$$\begin{array}{r} 2x - 2y = 0 \\ +\ -2x + 6y = 0 \\ \hline \Rightarrow \quad 0x + 4y = 0 \end{array} \Rightarrow y = 0$$

Then if $y = 0$

$$2x - 2(0) = 0$$

$$\Rightarrow 2x = 0$$

$$\Rightarrow x = 0$$

So we have a critical point at $(0,0)$

From the second partials test

$D > 0 \quad f_{xx} > 0 \Rightarrow$ minimum

$$f_{xx} = \frac{\partial}{\partial x} f_x = \frac{\partial}{\partial x}(2x - 2y) = 2$$

$$f_{yy} = \frac{\partial}{\partial y} f_y = \frac{\partial}{\partial y}(-2x + 6y) = 6$$

$$f_{xy} = \frac{\partial}{\partial y} f_x = \frac{\partial}{\partial y}(2x - 2y) = -2$$

$$D = f_{xx}f_{yy} - f_{xy}^2 = 2(6) - (-2)^2$$

$$= 12 - 4 = 8 > 0 \qquad f_{xx} = 2 > 0$$

$\boxed{(0,0) \text{ is a minimum}}$

25. (*10 points*) Find the extreme values of the function $f(x, y, z) = x + 2y^2 + y + 3z^2 - 2z$
Subject to the constraint $g(x, y, z) = x + y + z = 2$

with the given constraint we can use the method of Lagrange multipliers. we seek (x_0, y_0, z_0) and λ such that

$\nabla f(x_0, y_0, z_0) = \lambda \nabla g(x_0, y_0, z_0)$

and $g(x_0, y_0, z_0) = k$

here $k = 2$ $g(x,y,z) = x+y+z$

$f(x,y,z) = x + 2y^2 + y + 3z^2 - 2z$

$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$

$\nabla g = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle$

$\nabla f = \langle 1, 4y+1, 6z-2 \rangle$, $\nabla g = \langle 1, 1, 1 \rangle$

$\nabla f = \lambda \nabla g \Rightarrow \langle 1, 4y+1, 6z-2 \rangle = \lambda \langle 1, 1, 1 \rangle$

$\Rightarrow \langle 1, 4y+1, 6z-2 \rangle = \langle \lambda, \lambda, \lambda \rangle$

$\Rightarrow 1 = \lambda$ → This tells us $\lambda = 1$

Set like components equal to eachother

$4y+1 = \lambda$
$6z-2 = \lambda$

$\Rightarrow$

Solving for y, z

$4y+1 = 1 \rightarrow 4y = 0 \rightarrow \boxed{y = 0}$

$6z - 2 = 1 \rightarrow 6z = 3 \rightarrow \boxed{z = -1/2}$

Then we can use the constraint to find x

$x + y + z = 2 \quad y = 0, z = -1/2$

$\Rightarrow x + 0 - 1/2 = 2 \rightarrow x = 2 - 1/2 = 3/2$

$\boxed{x = 3/2}$

we have our extreme value at

$(3/2, 0, -1/2)$