

Sec 10.2 Derivatives and Integrals of Vector Functions.

$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$

We define the derivative of a vector function in the same way as for functions of one variable.

The derivative $\mathbf{r}'$ of a vector function $\mathbf{r}$ is defined in much the same way as for real-valued functions:

1

$$\frac{d\mathbf{r}}{dt} = \mathbf{r}'(t) = \lim_{h \rightarrow 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

We also have out limit theorem for vector functions

1

If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$, then

$$\lim_{t \rightarrow a} \mathbf{r}(t) = \langle \lim_{t \rightarrow a} f(t), \lim_{t \rightarrow a} g(t), \lim_{t \rightarrow a} h(t) \rangle$$

provided the limits of the component functions exist.

Applying this to the to the definition above gives us the statement:

$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$

$$\rightarrow \frac{d\mathbf{r}}{dt} = \lim_{h \rightarrow 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \langle \lim_{h \rightarrow 0} \frac{f(t+h) - f(t)}{h}, \lim_{h \rightarrow 0} \frac{g(t+h) - g(t)}{h}, \lim_{h \rightarrow 0} \frac{h(t+h) - h(t)}{h} \rangle$$

$$\lim_{h \rightarrow 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h} = \lim_{h \rightarrow 0} \frac{\langle f(t+h), g(t+h), h(t+h) \rangle - \langle f(t), g(t), h(t) \rangle}{h}$$

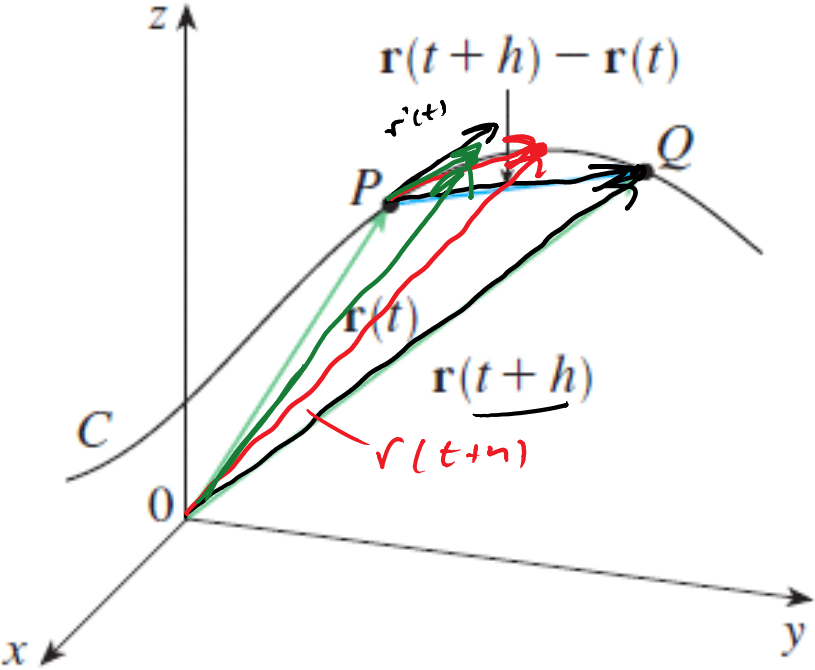
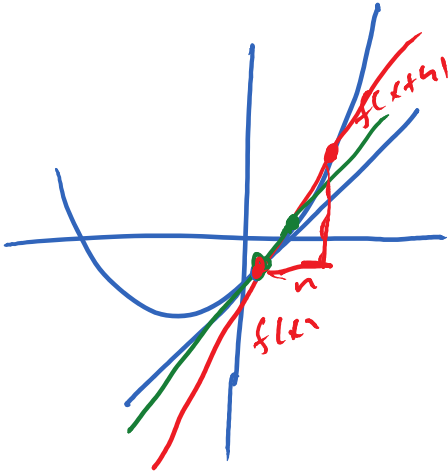
2

Theorem If $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f , g , and h are differentiable functions, then

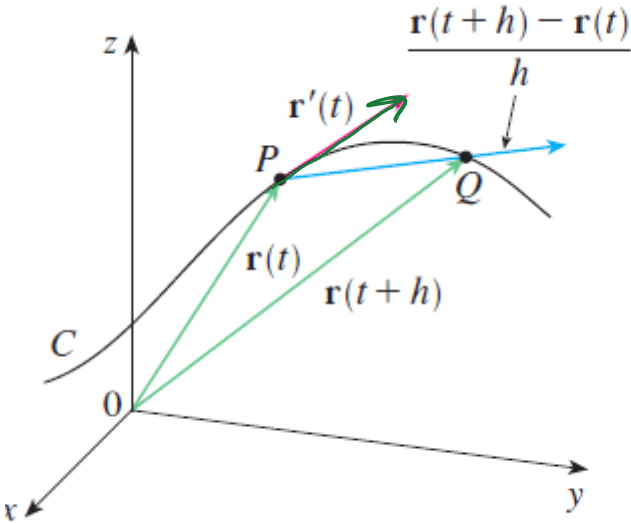
$$\mathbf{r}'(t) = \langle f'(t), g'(t), h'(t) \rangle = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$$

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Graphically, what we are doing?



(a) The secant vector



(b) The tangent vector

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Hw Example

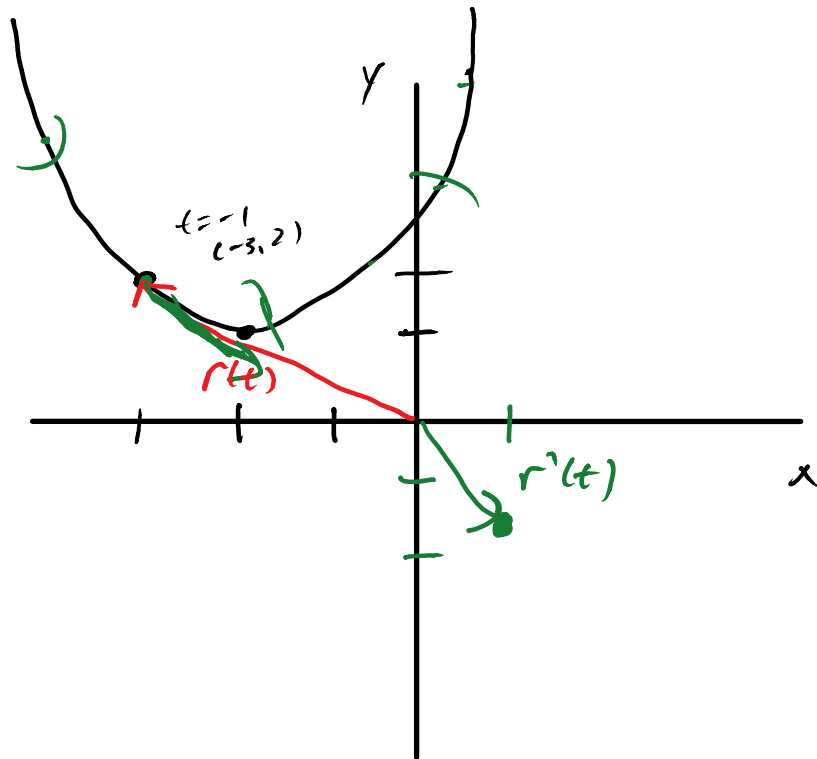
- Sketch the plane curve with the given vector equation.
- Find $\mathbf{r}'(t)$.
- Sketch the position vector $\mathbf{r}(t)$ and the tangent vector $\mathbf{r}'(t)$ for the given value of t .

3. $\mathbf{r}(t) = \langle t-2, t^2+1 \rangle, \quad t = -1$

t	(x, y)
-2	$(-4, 5)$ ←
-1	$(-3, 2)$ ←
0	$(-2, 1)$ ←
1	$(-1, 2)$ ←
2	$(0, 5)$ ←

$$x = t - 2 \quad y = t^2 + 1$$

$$x + 2 = t \quad y = (x + 2)^2 + 1$$



$$\vec{r}(t) = \langle t-2, t^2+1 \rangle$$

$$\vec{r}'(t) = \langle 1, 2t \rangle \rightarrow |\vec{r}'(t)| = \sqrt{1^2 + 4t^2}$$

$$\vec{r}'(-1) = \langle 1, -2 \rangle \Rightarrow |\vec{r}'(-1)| = \sqrt{1^2 + (-2)^2} = \sqrt{5}$$

Some times we are only interested in the direction of the tangent vector if that is the case we can find the **unit tangent vector**.

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

$$\vec{r}'(t) = \langle 1, 2t \rangle$$

$$|\vec{r}'(t)| = \sqrt{1^2 + (2t)^2} = \sqrt{1 + 4t^2}$$

$$\mathbf{T}(t) = \frac{1}{\sqrt{1+4t^2}} \langle 1, 2t \rangle = \left\langle \frac{1}{\sqrt{1+4t^2}}, \frac{2t}{\sqrt{1+4t^2}} \right\rangle$$

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HW Example:

21-24 Find parametric equations for the tangent line to the curve with the given parametric equations at the specified point.

23. $x = e^{-t} \cos t$, $y = e^{-t} \sin t$, $z = e^{-t}$; $(1, 0, 1)$

x, y, z

$$\mathbf{r}(t) = \langle e^{-t} \cos t, e^{-t} \sin t, e^{-t} \rangle$$

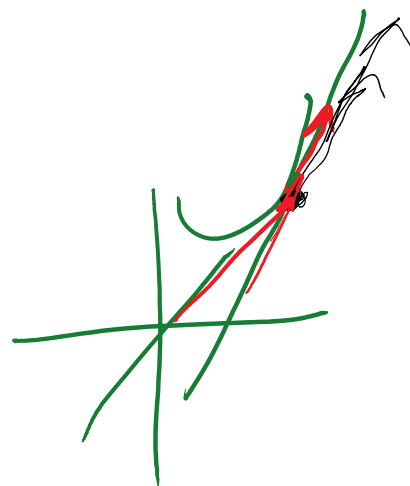
$$\mathbf{r}'(t) = \langle -e^{-t} \cos t - e^{-t} \sin t, -e^{-t} \sin t + e^{-t} \cos t, -e^{-t} \rangle$$

$$t = 0$$

$$\text{at } (x, y, z) = (1, 0, 1)$$

$$\mathbf{r}'(0) = \langle -e^{-0} \cos(0) - e^{-0} \sin(0), -e^{-0} \sin(0) + e^{-0} \cos(0), -e^{-0} \rangle$$

$$\mathbf{r}'(0) = \langle -1, 1, -1 \rangle$$



$$\frac{d}{dt} e^{-t} \cos t$$

$$-e^{-t} \cos t + e^{-t} (-\sin t)$$

$$e^{-t} \cos t = 1$$

$$e^{-t} \sin t = 0$$

$$e^{-t} = 1$$

$$x = x_0 + at \quad y = y_0 + bt \quad z = z_0 + ct$$

$$(x_0, y_0, z_0) = (1, 0, 1)$$

$$\langle a, b, c \rangle = \langle -1, 1, -1 \rangle$$

$$\boxed{x = 1 - t \quad y = 0 + t \quad z = 1 - t}$$

Differentiation Rules

The next theorem shows that the differentiation formulas for real-valued functions have their counterparts for vector-valued functions.

3 Theorem Suppose $\mathbf{u}$ and $\mathbf{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function. Then

1. $\frac{d}{dt}[\mathbf{u}(t) + \mathbf{v}(t)] = \mathbf{u}'(t) + \mathbf{v}'(t)$

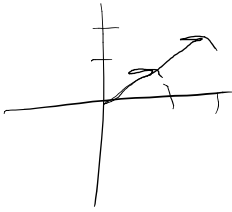
2. $\frac{d}{dt}[c\mathbf{u}(t)] = c\mathbf{u}'(t)$

3. $\frac{d}{dt}[f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t) \rightarrow \begin{matrix} f(t) \\ \uparrow \\ \sin(t) \end{matrix} \langle \cos(t), t^2 \rangle$
 $\begin{matrix} \cos(t) \end{matrix} \langle \cos(t), t^2 \rangle$

4. $\frac{d}{dt}[\mathbf{u}(t) \cdot \mathbf{v}(t)] = \mathbf{u}'(t) \cdot \mathbf{v}(t) + \mathbf{u}(t) \cdot \mathbf{v}'(t)$
 $\leftarrow \begin{matrix} \text{Scalar} \\ \text{Vector} \end{matrix}$

5. $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{v}(t)] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$

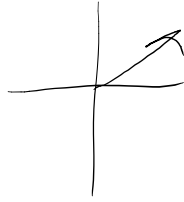
6. $\frac{d}{dt}[\mathbf{u}(f(t))] = f'(t)\mathbf{u}'(f(t))$ (Chain Rule)



Scalar

2

$2 \langle 1, 1 \rangle$
 $\langle 2, 2 \rangle$



Vector

$\langle 1, 2, 3 \rangle$

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Integrals

The **definite integral** of a continuous vector function $\mathbf{r}(t)$ can be defined in much the same way as for real-valued functions except that the integral is a vector. But then we can express the integral of $\mathbf{r}$ in terms of the integrals of its component functions $f, g,$ and h as follows. (We use the notation of Chapter 5.)

$$\int_a^b \mathbf{r}(t) dt = \lim_{n \rightarrow \infty} \sum_{i=1}^n \mathbf{r}(t_i^*) \Delta t$$
$$= \lim_{n \rightarrow \infty} \left[\left(\sum_{i=1}^n f(t_i^*) \Delta t \right) \mathbf{i} + \left(\sum_{i=1}^n g(t_i^*) \Delta t \right) \mathbf{j} + \left(\sum_{i=1}^n h(t_i^*) \Delta t \right) \mathbf{k} \right]$$

and so

$$\int_a^b \mathbf{r}(t) dt = \left(\int_a^b f(t) dt \right) \mathbf{i} + \left(\int_a^b g(t) dt \right) \mathbf{j} + \left(\int_a^b h(t) dt \right) \mathbf{k}$$

HW Example

$\langle f(t), g(t), h(t) \rangle$

35. $\int_0^{\pi/2} (3 \sin^2 t \cos t \mathbf{i} + 3 \sin t \cos^2 t \mathbf{j} + 2 \sin t \cos t \mathbf{k}) dt$

$\int \frac{1}{2} \sin(2t) = \frac{1}{2} \sin(2t)$
 $\sin(2x) = 2 \cos(x) \sin(x)$

$= \int_0^{\pi/2} 3 \sin^2(t) \cos(t) dt + \int_0^{\pi/2} 3 \sin t \cos^2 t dt + \int_0^{\pi/2} 2 \sin t \cos t dt$

$u = \sin(t) \quad du = \cos(t) dt$
 $u = \cos(t) \quad du = -\sin(t) dt$
 $u = \sin(t) \quad du = \cos(t) dt$

$\int 3u^2 \hat{i} + \int -3u^2 dt \hat{j} + \int 2u du \hat{k}$

$\langle \int 3u^2 dt, -\int 3u^2 dt, \int 2u du \rangle$

$\langle \frac{3u^3}{3} - \frac{3u^3}{3}, \frac{2u^2}{2} \rangle = \langle u^3 - u^3, u^2 \rangle$

$\langle \sin^3(\pi/2) - \cos^3(\pi/2), \sin^2(\pi/2) \rangle = \langle 1 - (-1), 1 \rangle = \langle 2, 1 \rangle$

$\langle \sin^3(0) - \cos^3(0), \sin^2(0) \rangle = \langle 0 - 1, 0 \rangle = \langle -1, 0 \rangle$

$$\langle 1, 1, 1 \rangle \xrightarrow{0} - \langle \cancel{\sin^3(0)}, -\cancel{\cos^3(0)}, \cancel{\sin^2(0)} \rangle$$