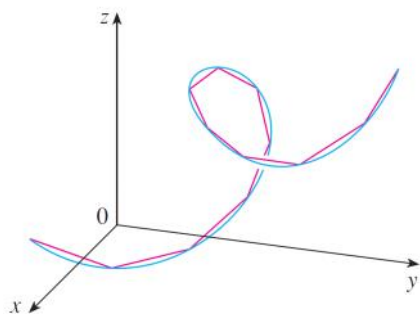


Sec 10.3 Arc Length and Curvature

You might recall from sec 6.4 that we used the distance formula with success approximations between points on a curve to derive a formula which gave us the length of a curve for a function in the x-y plane.

It turns out that we can do the same thing with curves in 3-d.



2

$$\begin{aligned} L &= \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2 + [h'(t)]^2} \, dt \\ &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} \, dt \end{aligned}$$

$$L = \int_a^b |\mathbf{r}'(t)| \, dt$$

Sec 10.3 Arc Length and Curvature

HW Example Find the length of the curve given by:

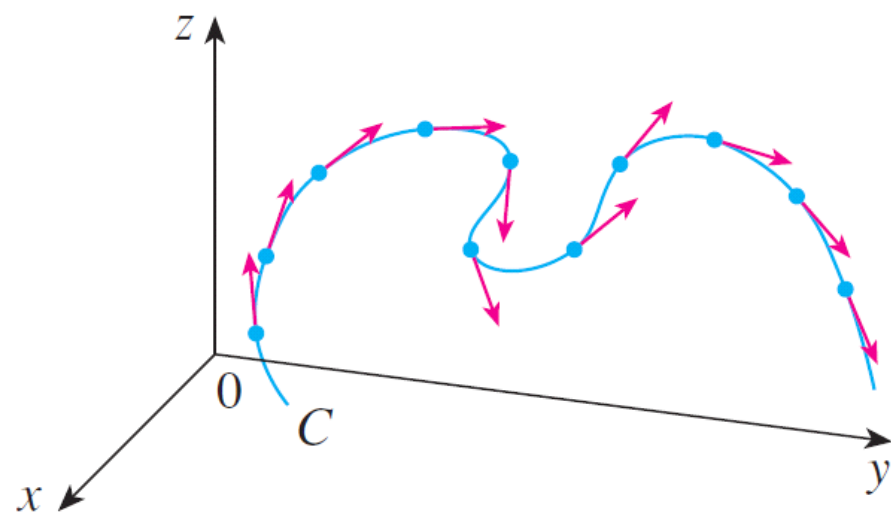
1. $\mathbf{r}(t) = \langle 2 \sin t, 5t, 2 \cos t \rangle, \quad -10 \leq t \leq 10$

Sec 10.3 Arc Length and Curvature

Curvature. Joke break: Best pick up line to use in calculus class: Can I be your derivative? I want to lie tangent to your curves....

Anyway...

We have talked about the unit tangent vector to a given parameterized curve defined by the derivative of the vector function which represents that curve.



$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

Consider the rate at which this unit vector is changing with respect to arc length. Think of it as how quickly the tangent vector is changing direction as we move some distance on the curve. We can represent it with $\left|\frac{dT}{ds}\right|$

If the direction of the unit tangent vector changes very little over a distance:



If the direction of the unit tangent vector changes quickly over a distance



In this way we can use the rate that the unit vector changes with arc length as a measure of curvature!

Sec 10.3 Arc Length and Curvature

8 Definition The **curvature** of a curve is

$$\kappa = \left| \frac{d\mathbf{T}}{ds} \right|$$

where $\mathbf{T}$ is the unit tangent vector.

Where:

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$$

Of course this can be difficult to calculate as most of the time our curves are not given in terms of arc length s but instead parameterized, usually by t .

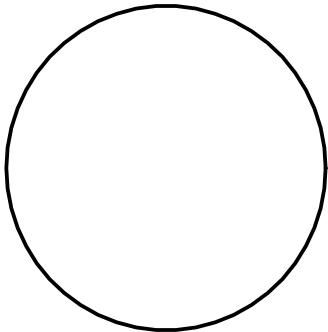
However with some clever tricks we can re-write this formula. Recall that from the chain rule we can say:

$$\frac{d\mathbf{T}}{dt} = \frac{d\mathbf{T}}{ds} \frac{ds}{dt}$$

9

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$$

Let's quickly calculate the curvature of a circle. A circle centered at the origin can be parameterized by:
 $\mathbf{r}(t) = \langle r\cos(t), r\sin(t) \rangle$



Sec 10.3 Arc Length and Curvature

Hw Example

25. Find the curvature of $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ at the point $(1, 1, 1)$.

Sec 10.3 Arc Length and Curvature

10 Theorem The curvature of the curve given by the vector function $\mathbf{r}$ is

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

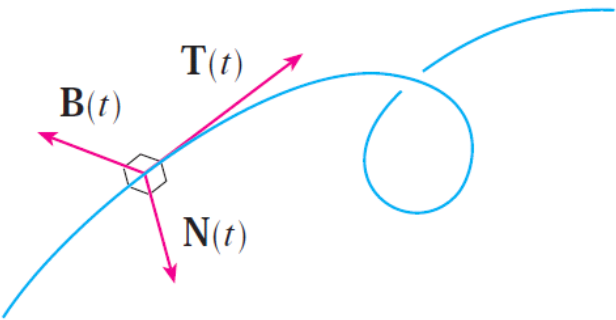
HW Example:

21–23 Use Theorem 10 to find the curvature.

23. $\mathbf{r}(t) = 3t\mathbf{i} + 4 \sin t\mathbf{j} + 4 \cos t\mathbf{k}$

Sec 10.3 Arc Length and Curvature

Other important vectors we need to understand curves in 3d are the Normal (**N(t)**) and Binormal (**B(t)**) vectors.



The Unit Normal Vector is defined to be:

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

It tells us which direction a curve is turning. This is because it is the derivative of the unit tangent vector. The unit tangent vector tells where the curve is headed. The derivative of the unit tangent vector tells us where the unit tangent vector is headed. For the physicist out there think of these as velocity and acceleration.

The Binormal vector is simply defined as:

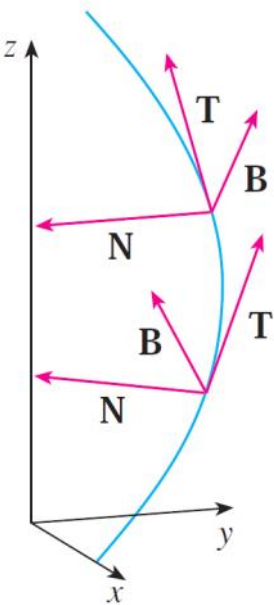
$$\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$$

Since this is a cross product we know it is orthogonal to both **T(t)** and **N(t)**

Hw Example:

45–46 Find the vectors **T**, **N**, and **B** at the given point.

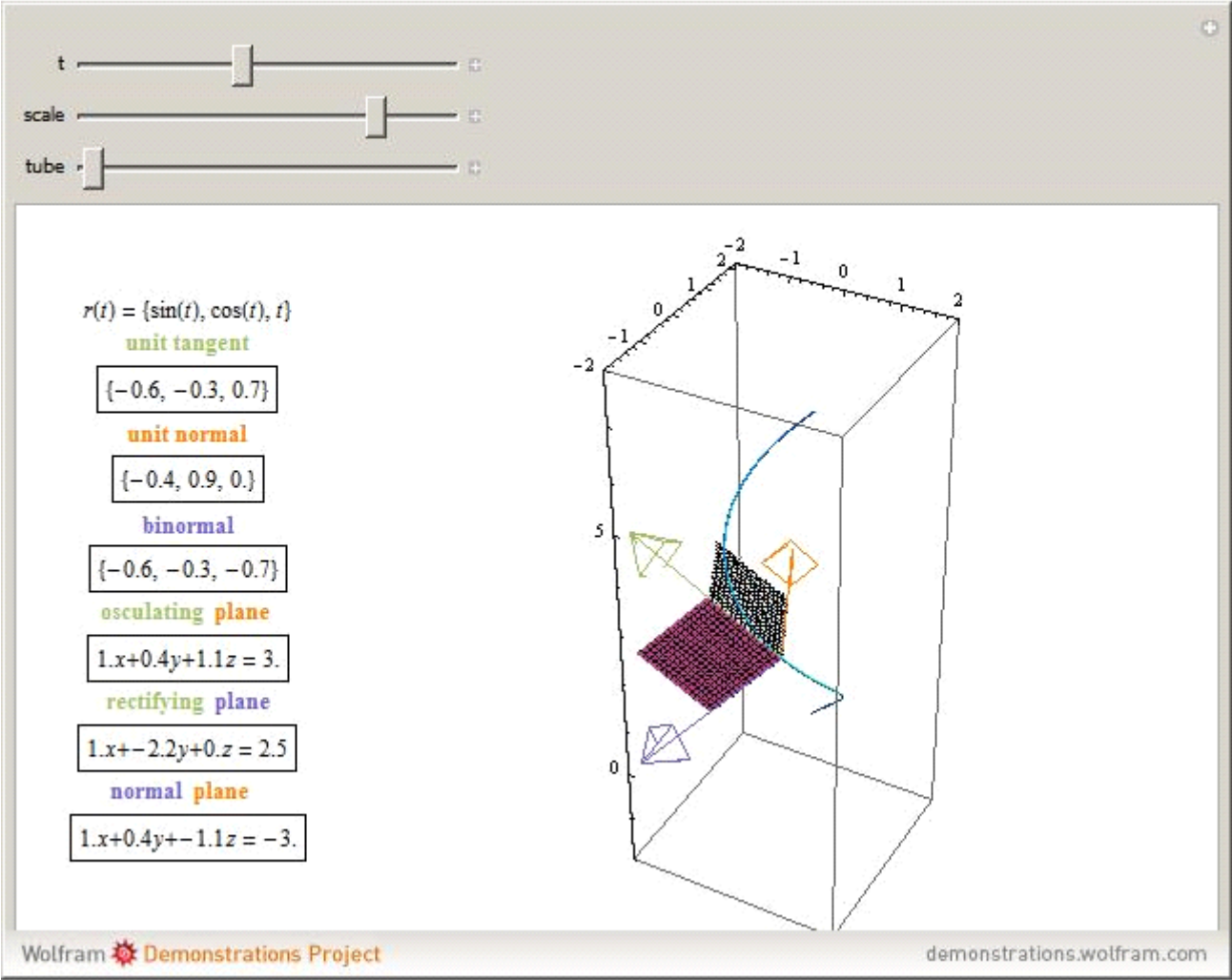
45. $\mathbf{r}(t) = \langle t^2, \frac{2}{3}t^3, t \rangle, \quad (1, \frac{2}{3}, 1)$



The Tangent, Normal, and Binormal vectors moving up a section of a spiral. <http://jan.ucc.nau.edu/~jws8/137.2001.1/sect10.3/TNBmovie.gif>

Sec 10.3 Arc Length and Curvature

The planes defined by the Normal and Binormal vectors on a curve are tangent to the curve and called the normal planes. The plane defined by the unit Tangent vector is called the osculating plane.



<http://demonstrations.wolfram.com/FrenetFrame/>

Sec 10.3 Arc Length and Curvature

HW Example

47–48 Find equations of the normal plane and osculating plane of the curve at the given point.

47. $x = 2 \sin 3t$, $y = t$, $z = 2 \cos 3t$; $(0, \pi, -2)$