

Sec 7.4 (Exponential Growth and Decay)

When We first talked about differential equations we came up with the idea that populations will grow at a rate proportional to the actual population we wrote this as

$\frac{dx}{dt} = kx$ We also guessed a solution of $x = e^{kt}$ And verified that this is indeed a solution to the DE. In general

we can use this model for any quantity whose rate of change is proportional to itself. There are many things which behave in this fashion.

Some of the most common are:

Populations of Bacteria

Continuously compounded interest

Radio active decay

The Temperature of an object left to cool.

We will do examples of these but first lets solve our DE using separation of variables

Suppose

$\frac{dx}{dt} = kx$ $x(0) = x_0$ Find a solution to the IVP.

$$\frac{dx}{dt} = kx$$

$$dx = kx dt$$

$$\frac{dx}{x} = k dt$$

$$\int \frac{1}{x} dx = \int k dt$$

$$\ln(x) = kt + C$$

$$e^{\ln(x)} = e^{kt+C}$$

$$x = e^{kt+C}$$

$$x = e^{kt} e^C$$

$$x = \tilde{C} e^{kt}$$

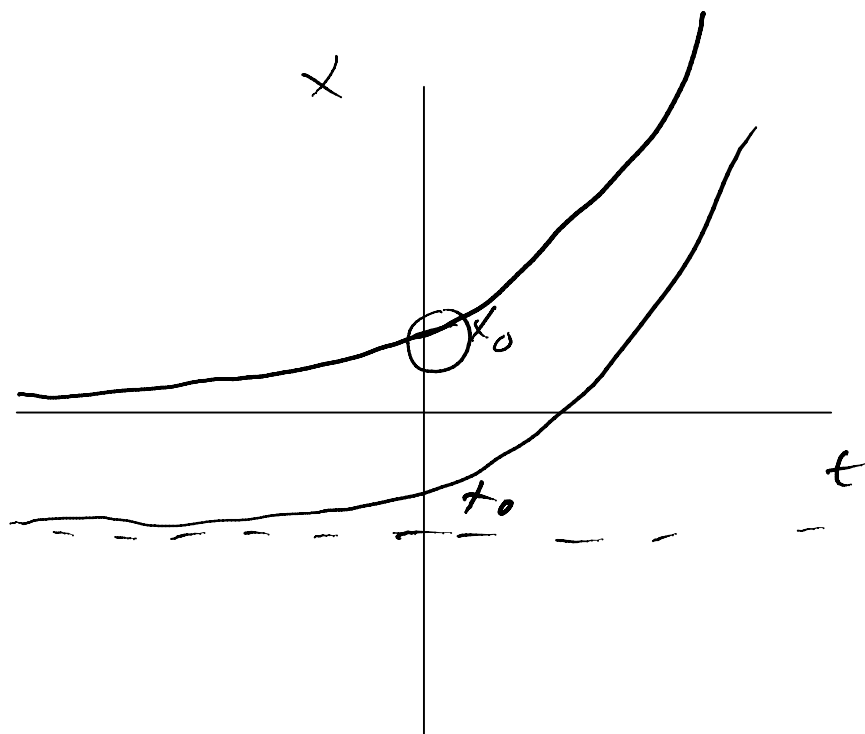
$$x_0 = \tilde{C}$$

$$x(t) = x_0 e^{kt}$$

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Let's Examine our general solution $x = x_0 e^{kt}$

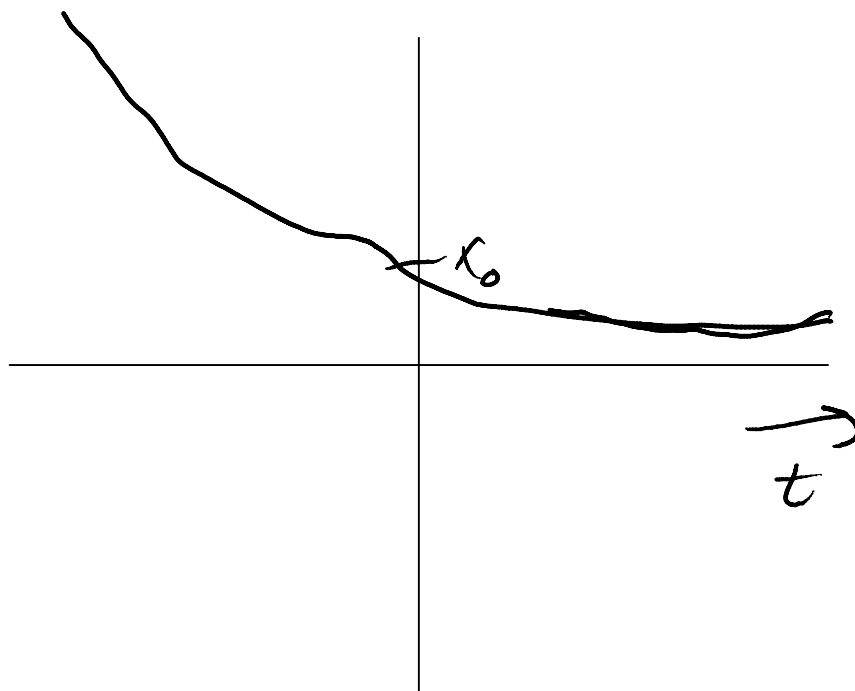
What does it mean if $k > 0$



What does it mean if $k < 0$

$$k = -2$$

$$x_0 e^{-2t}$$



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Population growth

$$\text{world } X = X_0 e^{kt}$$

Suppose the population in the united states in 1900 was 1650 million. In 1920 the population was 1860 million. Using this information what will be population be in 2000? Assume the population grows exponentially

$$P(t) = P_0 e^{kt}$$

$$P(t) = 1650 e^{kt}$$

$$P(20) = 1650 e^{k(20)}$$

$$1860 = 1650 e^{20k}$$

$$\frac{1860}{1650} = e^{20k}$$

$$\ln\left(\frac{1860}{1650}\right) = \ln(e^{20k})$$

$$\ln\left(\frac{1860}{1650}\right) = 20k$$

$$k = \frac{1}{20} \ln\left(\frac{1860}{1650}\right)$$

$$\frac{1}{20} \ln\left(\frac{1860}{1650}\right)t$$

$$P(t) = 1650 e$$

$$k = .00599$$

$$= .006$$

$$P(t) = 1650 e^{.006t}$$

$$P(100) = 1650 e^{.006(100)}$$

$$P(100) = 1650 e^{.6}$$

$$= \boxed{3006.5 \text{ million}}$$

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Radio active decay: For radio active decay the rate of change of the amount of mass of a given radio active element (m) is proportional to the amount of the element. This can be written as:

$$\frac{dm}{dt} = -km \text{ where } k > 0 \text{ and the - sign gives decay.}$$

$$m = m_0 e^{-kt}$$

~~The half life of C14 is 5730 (that is after that time half of the mass of C14 remains from the original sample)~~

When living things are alive they make C14 naturally , however when they die the C14 decays into C12. If you find a piece of wood which has a C14 to C12 ratio of $1/4$ how long ago did the tree it came from die?

$$m = m_0 e^{-kt}$$

$$\rightarrow \frac{m}{m_0} = e^{-kt}$$

$$\frac{1}{5} = e^{-kt}$$

$$-0.000121 t$$

$$\frac{1}{5} = e^{-kt}$$

$$\ln\left(\frac{1}{5}\right) = \ln(e^{-0.000121 t})$$

$$\ln\left(\frac{1}{5}\right) = -0.000121 t$$

$$t = \frac{\ln(1/5)}{-0.000121}$$

$$t = 13,301.1 \text{ years}$$

$$\frac{C_{14}}{C_{12}} = \frac{1}{4}$$

10 g

$$\frac{C_{14}}{C_{12}} = \frac{1}{4} \quad \frac{2}{8}$$

5g C14

$$\frac{1}{2} = e^{-k(5730)}$$

$$\ln\left(\frac{1}{2}\right) = \ln(e^{-k(5730)})$$

$$\ln\left(\frac{1}{2}\right) = -k(5730)$$

$$-k = \frac{\ln(1/2)}{5730} = -0.000121$$

$$k = 0.000121$$

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Newton's Law of cooling: Newton's law of cooling states that the rate of change of temperature with time is proportional to the difference in current temperature of an object and the ambient temperature. That is to say:

$$\frac{dT}{dt} = k(T - T_a)$$

After a really tough exam given to his 1320 class Christian is found dead in his office. :(His body temperature was measured at time of discovery to be 80 F 30 minutes later his body temperature was measured at 75 F. How long before the time of discovery was he murdered given that the ambient temperature in his office was a constant 70 F.

$$\frac{dT}{dt} = k(T - T_a)$$

$$\int \frac{dT}{(T - T_a)} = \int k dt$$

$$u = T - T_a$$

$$du = dt$$

$$\int \frac{1}{u} du = kt + C$$

$$\ln(T - T_a) = kt + C$$

$$T - T_a = e^{kt + C}$$

$$T - T_a = e^{kt + C}$$

$$T - T_a = \tilde{C} e^{kt}$$

$$T = \tilde{C} e^{kt} + T_a$$

$$T(0) = \tilde{C} e^0 + 70$$

$$80 = \tilde{C} + 70$$

$$80 - 70 = \tilde{C} = 10$$

$$T(t) = (T_0 - T_a) e^{k(t)} + T_a$$

$$T(t) = 10 e^{kt} + 70$$

$$75 = 10 e^{k(1/2)} + 70$$

$$5 = 10 e^{k(1/2)}$$

$$\frac{5}{10} = e^{k/2}$$

$$1/2 = e^{k/2}$$

$$\ln(1/2) = \ln(e^{k/2})$$

$$\ln(1/2) = k/2$$

$$k = 2 \ln(1/2)$$

$$T(t) = 10 e^{2 \ln(1/2)t} + 70$$

$$k = -1.38629$$

$$|k| \approx -1.4$$

$$80 - 70 = \hat{c} = 10$$

$$k = 1.4$$

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$$T(t) = 10e^{-1.4t} + 70$$

$$98.6^{\circ} = 10e^{-1.4t} + 70$$

$$28.6^{\circ} = 10e^{-1.4t}$$

$$2.86 = e^{-1.4t}$$

$$\ln(2.86) = \ln(e^{-1.4t})$$

$$\ln(2.86) = -1.4t$$

$$t = \frac{\ln(2.86)}{-1.4}$$

$$t = -.75$$

$t \approx 45$ min Before
I was found.

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When interest in a bank account is applied continuously the rate of change of the amount of money in the account is proportional to the amount in the account. I.e.

$\frac{dA}{dt} = rA$ where r is the interest rate. If I am lucky enough to find an account that gives 5% interest compounded continuously and I invest \$100 dollars when will I have \$500 in my account.

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