

Sec 8.2 (Series)

What is a series? A series is a sum of the terms of a sequence.

Ex. Consider the sequence $a_n = \frac{1}{2^n}$. The series for this sequence would be defined as

$$\sum_{n=0}^N a_n = \sum_{n=0}^N \frac{1}{2^n} = \frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^N} = \left(\frac{1 + (1/2)^{N+1}}{1 - 1/2} \right) = S_N$$

$\uparrow \quad \uparrow \quad \uparrow \quad \uparrow \quad \uparrow$
 $n=0 \quad n=1 \quad n=2 \quad n=3 \quad n=N$

We can also have infinite series where we add infinitely many terms:

$$\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{2^0} + \frac{1}{2^1} + \frac{1}{2^2} + \frac{1}{2^3} + \dots$$

$\uparrow \quad \uparrow \quad \uparrow \quad \uparrow$
 $n=0 \quad n=1 \quad n=2 \quad n=3$

Ex Consider sequence $a_n = n^2 + 1$ Write the sum of the first 10 terms in series notation. What is the sum of the first 10 terms?

$n=1 \quad n=2 \quad n=3 \quad n=4 \quad n=10$

$$(1^2+1) + (2^2+1) + (3^2+1) + (4^2+1) + \dots + (10^2+1)$$

$$\sum_{n=1}^{10} n^2 + 1$$

$2 + 5 + 10 + 17 + 26 + 37 + 50 + 65 + 82 + 101 = ?$

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The partial sum of a series is defined at the sum of the first n terms of that series. Often denoted S_n
For an infinite series we can take the limit as n goes to infinity...

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Definition Given a series $\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \dots$, let s_n denote its n th partial sum:

$$s_n = \sum_{i=1}^n a_i = a_1 + a_2 + \dots + a_n$$

If the sequence $\{s_n\}$ is convergent and $\lim_{n \rightarrow \infty} s_n = s$ exists as a real number, then the series $\sum a_n$ is called **convergent** and we write

$$a_1 + a_2 + \dots + a_n + \dots \quad \text{or} \quad \sum_{n=1}^{\infty} a_n = s$$

The number s is called the **sum** of the series. If the sequence $\{s_n\}$ is divergent, then the series is called **divergent**.

How is it that we can add infinitely many terms and get something finite?

$$a_n = \frac{1}{10^n}$$
$$a_0 = 1 \quad a_1 = .1 \quad a_2 = .01$$
$$a_3 = .001$$
$$\sum_{n=0}^{\infty} \frac{1}{10^n} = 1 + .1 + .01 + .001 + .0001 + \dots$$
$$= 1.111111111$$

$$\sum_{n=0}^{\infty} (-1)^n = 1 - 1 + 1 - 1 + \dots$$

$$S_0 = 1$$
$$S_1 = 0$$
$$S_2 = 1$$

$$S_3 = 0$$
$$S_4 = 1$$

$$S_n =$$

Sec 8.2 (Series)

The Geometric Series: A Geometric series is the sum of the terms of a geometric sequence one particularly useful in calculus.

The Partial Sum of a Geometric Series. Consider the geometric sequence $a_n = a_1 r^n$. Find an expression for the partial sum of the first n terms.

$$\sum_{i=0}^n a_1 r^i = a_1 (r^0 + r^1 + r^2 + r^3 + \dots + r^n)$$

$$S = a_1 (r^0 + r^1 + r^2 + r^3 + \dots + r^n)$$

$$(1-r)S = a_1 (1-r)(r^0 + r^1 + r^2 + r^3 + \dots + r^n)$$

$\begin{cases} r^2 = r^2 \\ r^3 = r^4 \end{cases}$

$$(1-r)S = a_1 ((r^0 + r^1 + r^2 + r^3 + \dots + r^n) - r(r^0 + r^1 + r^2 + r^3 + \dots + r^n))$$

$$(1-r)S = a_1 (r^0 + \cancel{r^1} + \cancel{r^2} + \cancel{r^3} + \dots + \cancel{r^n} - \cancel{r^1} - \cancel{r^2} - \cancel{r^3} - \cancel{r^4} + \dots - \cancel{r^n} - r^{n+1})$$

$$(1-r)S = a_1 (1 - r^{n+1})$$

$$S_n = \frac{a_1 (1 - r^{n+1})}{1 - r}$$

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Infinite sum of a geometric series:

Consider the partial sum of a geometric series :

$$S_n = \sum_{i=0}^n a_1 r^i = \frac{a_1(1-r^{n+1})}{1-r}$$

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} a_1 \frac{(1-r^{n+1})}{1-r}$$

$$= a_1 \left(\frac{1 - \lim_{n \rightarrow \infty} r^{n+1}}{1-r} \right) =$$

$|r| < 1$

if $|r| < 1$

$$\lim_{n \rightarrow \infty} r^{n+1} = 0$$

$r = 1/2$

$$\lim_{n \rightarrow \infty} \left(\frac{1}{2}\right)^{n+1} = \lim_{n \rightarrow \infty} \frac{1}{2^{n+1}}$$

$$\Rightarrow a_1 \left(\frac{1 - \lim_{n \rightarrow \infty} r^{n+1}}{1-r} \right) = a_1 \left(\frac{1-0}{1-r} \right) = \boxed{a_1 \frac{1}{1-r}}$$

if $|r| < 1$

$$= \frac{a_1}{1-r}$$

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$$\sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \dots$$

is convergent if $|r| < 1$ and its sum is

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r} \quad |r| < 1$$

If $|r| \geq 1$, the geometric series is divergent.

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Ex: Find

$$\sum_{i=0}^{\infty} \frac{1}{2^n} =$$

$$\sum_{i=1}^{\infty} \frac{1}{2^{n-1}}$$

$$a_n = \frac{1}{2^n} = \left(\frac{1}{2}\right)^n$$

$$a_n = a_1 r^n$$

$$a_1 = 1$$

$$\sum_{i=0}^{\infty} a_1 r^n = \frac{a_1}{1-r}$$

$$a_1 = 1 \quad r = 1/2$$

$$\sum_{i=0}^{\infty} \frac{1}{2^n} = \frac{1}{1-1/2} = \frac{1}{1/2} = \boxed{2}$$

$$\sum_{i=2}^{\infty} \frac{1}{2^i} =$$

$$\sum_{i=0}^{\infty} \frac{1}{2^i} = \underbrace{1 + 1/2 + 1/4 + 1/8 + 1/16 + \dots}_{1-1/2} = \frac{1}{1-1/2}$$

$$\sum_{i=2}^{\infty} \frac{1}{2^i} = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots$$

$$= \frac{1}{1-1/2} - 1 - 1/2 = 2 - 1 - 1/2 = 1/2$$

$$\sum_{i=N}^{\infty} a_1 r^i = \frac{a_1}{1-r} - \left(\frac{a_1 (1-r^{N+1})}{1-r} \right)$$

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Telescoping Series

Find the sum of the series...

$$\sum_{i=1}^{\infty} \frac{1}{i(i+1)}$$

$$\frac{1}{1(2)} + \frac{1}{2(3)} + \frac{1}{3(4)} + \dots$$

$$\cancel{i(i+1)} \frac{1}{\cancel{i(i+1)}} = \left(\frac{A}{i} + \frac{B}{i+1} \right) \cancel{i(i+1)}$$

$$1 = A(i+1) + Bi$$

$$i=0$$

$$1 = A(0+1) + 0 \Rightarrow A=1$$

$$i=-1$$

$$1 = A(0) - B \quad 1 = -B \quad B = -1$$

$$\sum_{i=1}^{\infty} \frac{1}{i(i+1)} = \sum_{i=1}^{\infty} \frac{1}{i} - \frac{1}{i+1}$$

$$\begin{array}{cccccc} i=1 & & i=2 & & i=3 & & i=4 & & i=5 \\ \left(\frac{1}{1} - \frac{1}{2} \right) & + & \left(\frac{1}{2} - \frac{1}{3} \right) & + & \left(\frac{1}{3} - \frac{1}{4} \right) & + & \left(\frac{1}{4} - \frac{1}{5} \right) & + & \left(\frac{1}{5} - \frac{1}{6} \right) \end{array}$$

$$+ \dots + = 1$$

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The Harmonic series :

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{10000}$$

Does this series converge? Lets consider some partial sums in particular s_2, s_4, s_8, s_{16}

$$s_2 = 1 + \frac{1}{2}$$
$$s_4 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) = 1 + \frac{2}{2}$$
$$s_8 = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right)$$
$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8}\right) =$$
$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{3}{2} = 2.5$$
$$s_{16} = 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \dots + \frac{1}{8}\right) + \left(\frac{1}{9} + \dots + \frac{1}{16}\right)$$
$$> 1 + \frac{1}{2} + \left(\frac{1}{4} + \frac{1}{4}\right) + \left(\frac{1}{8} + \dots + \frac{1}{8}\right) + \left(\frac{1}{16} + \dots + \frac{1}{16}\right)$$
$$= 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1 + \frac{4}{2} = 3$$

$$1 + \frac{1}{2} + \frac{1}{2} = 2$$
$$s_2 = 1.5$$
$$s_4 > 2$$
$$s_8 > 2.5$$
$$s_{16} > 3$$

Similarly, $s_{32} > 1 + \frac{5}{2}, s_{64} > 1 + \frac{6}{2}$, and in general

$$s_{2^n} > 1 + \frac{n}{2}$$
$$s_{2^n} > 1 + \frac{n}{2}$$

This shows that $s_{2^n} \rightarrow \infty$ as $n \rightarrow \infty$ and so $\{s_n\}$ is divergent. Therefore the harmonic series diverges.

$$\lim_{n \rightarrow \infty} s_{2^n} \geq \lim_{n \rightarrow \infty} 1 + \frac{n}{2} = \infty$$

The Harmonic series will be very useful in your near future to show that other series diverge!

Ex Show that the series diverges

$$\sum_{i=1}^{\infty} \frac{n^2 + 1}{3n^3}$$

$$a_n = \frac{n^2 + 1}{3n^3}$$

$$\lim_{n \rightarrow \infty} a_n = 0$$

$$a_n = \frac{n^2 + 1}{3n^3} > \frac{n^2}{3n^3} = \frac{1}{3n}$$
$$\sum_{n=1}^{\infty} \frac{n^2 + 1}{3n^3} > \sum_{n=1}^{\infty} \frac{1}{3n} = \frac{1}{3} \sum_{n=1}^{\infty} \frac{1}{n}$$

$$\infty$$
$$\infty$$

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A tricky Theorem about convergence.

$$\sum_{n=1}^{\infty} a_n$$

$$\lim_{n \rightarrow \infty} a_n = 0$$

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Theorem If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n \rightarrow \infty} a_n = 0$.



$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

$$\sum \frac{1}{n} = \infty$$

$$\sum a_n = C$$

$$\lim_{n \rightarrow \infty} a_n = 0$$

Ex.
 $\sum_{i=0}^{\infty} \frac{1}{2^n}$

$$= \sum_{i=0}^{\infty} \left(\frac{1}{2}\right)^n$$

$$r = 1/2$$

$$|r| < 1$$

$$\lim_{n \rightarrow \infty} \frac{1}{2^n} = 0$$

What about the Harmonic series?

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

$$= \infty$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} = 0$$

Don't be fooled! Just because the limit of the sequence is zero does NOT mean that the series will converge, rather we can say that if the limit of the sequence is not zero the series diverges.

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The Test for Divergence If $\lim_{n \rightarrow \infty} a_n$ does not exist or if $\lim_{n \rightarrow \infty} a_n \neq 0$, then the series $\sum_{n=1}^{\infty} a_n$ is divergent.



Ex Show that the series diverges

$$\sum_{n=0}^{\infty} \frac{2n^2 + 1}{3n^2 + n}$$



diverges

$$\lim_{n \rightarrow \infty} \frac{2n^2 + 1}{3n^2 + n} = \frac{\infty}{\infty}$$

$$\lim_{n \rightarrow \infty} \frac{4n}{6n + 1} = \frac{\infty}{\infty}$$

$$= \lim_{n \rightarrow \infty} \frac{4}{6} = \frac{4}{6} = \frac{2}{3} \neq 0$$

8 Theorem If $\sum a_n$ and $\sum b_n$ are convergent series, then so are the series $\sum ca_n$ (where c is a constant), $\sum (a_n + b_n)$, and $\sum (a_n - b_n)$, and

$(i) \sum_{n=1}^{\infty} ca_n = c \sum_{n=1}^{\infty} a_n$

$(ii) \sum_{n=1}^{\infty} (a_n + b_n) = \sum_{n=1}^{\infty} a_n + \sum_{n=1}^{\infty} b_n$

$(iii) \sum_{n=1}^{\infty} (a_n - b_n) = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^{\infty} b_n$

Ex Find the sum if possible.

$$\sum_{n=1}^{\infty} \frac{1}{e^n} + \frac{1}{n(n+1)}$$
$$= \sum_{n=1}^{\infty} \frac{1}{e^n} + \sum_{n=1}^{\infty} \frac{1}{n(n+1)}$$
$$= \sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n + 1$$

Geometric $r = 1/e = 2.718281828459045 \dots$
 $|1/e| < 1$

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$
$$= \sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n + 1$$
$$= \sum_{n=0}^{\infty} \left(\frac{1}{e}\right)^n - \left(\frac{1}{e}\right)^0 + 1$$
$$\frac{1}{1-\frac{1}{e}} - 1 + 1 = \frac{1}{1-\frac{1}{e}}$$

$$\sum_{n=0}^{\infty} \left(\frac{1}{e}\right)^n = \left(\frac{1}{e}\right)^0 + \left(\frac{1}{e}\right)^1 + \left(\frac{1}{e}\right)^2 + \dots$$