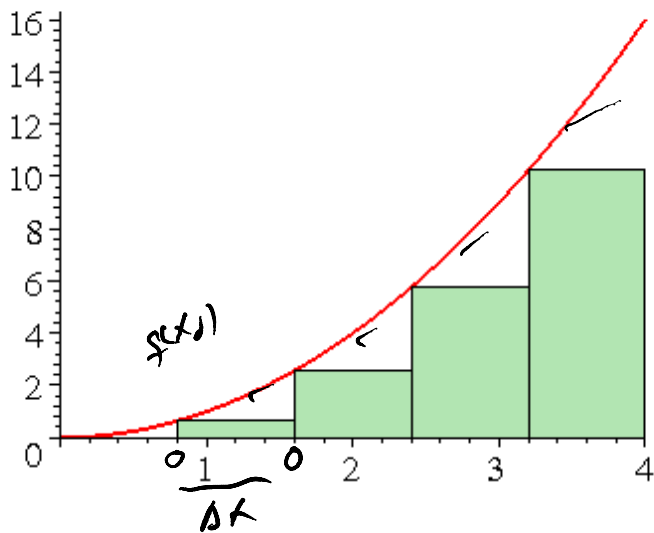


Sec 8.3 Integral and Comparison tests

Series are very important in engineering as they are often used to model real world problems. It is useful to have methods for determining whether or not a series converges.

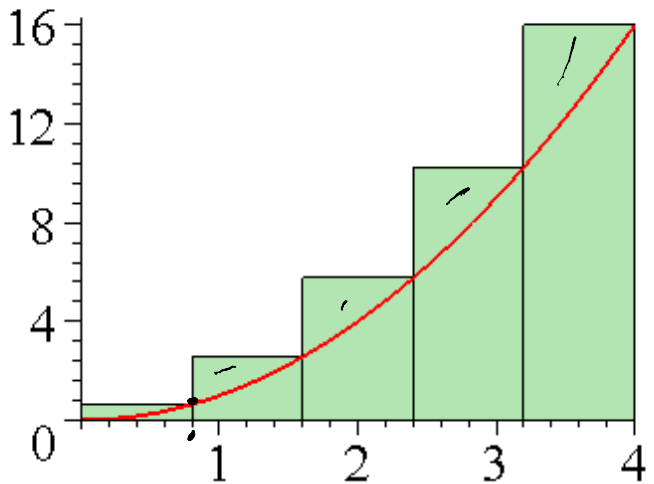
One such test is the integral test. You might recall from your first semester of calculus the left and right Riemann sums.

Left-Hand Approximating Rectangles



The left hand sum underestimates the area

Right-Hand Approximating Rectangles



The right hand sum over estimates the area

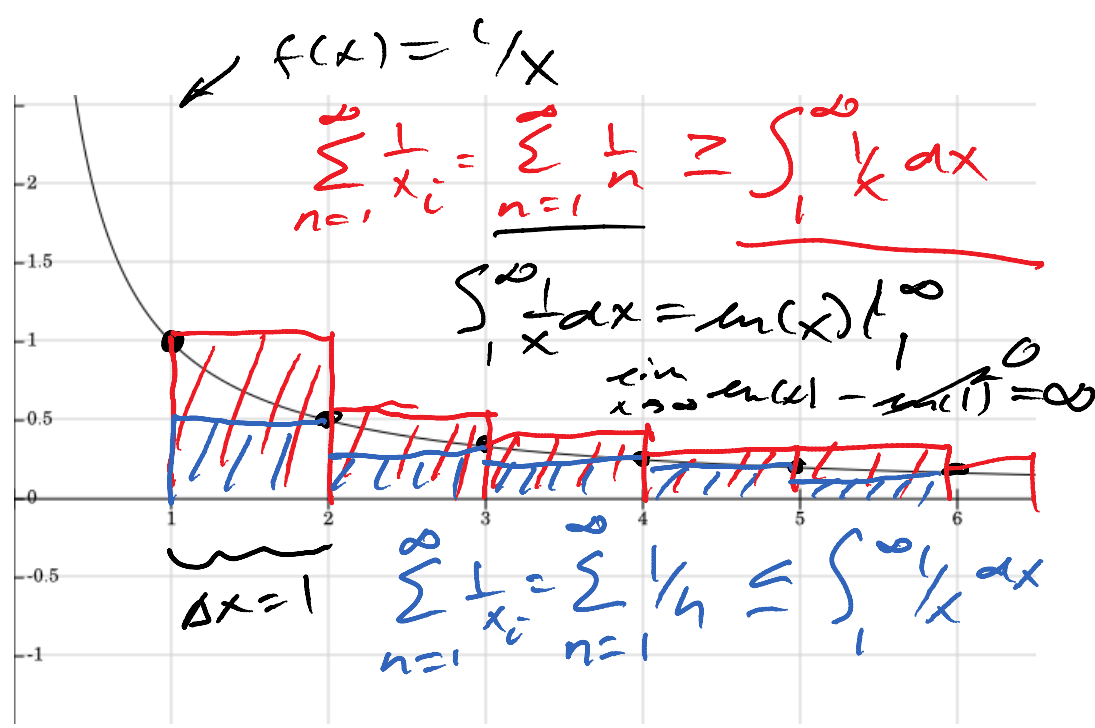
Consider $\sum_{n=1}^{\infty} \frac{1}{n}$

$\downarrow$

$\sum_{n=1}^{\infty} \frac{1}{x_i} \Delta x$

$\sum_{n=1}^{\infty} \frac{1}{x_i}$

$x_i = 1, 2, \dots$



So the with $\Delta x = 1$ the Riemann sums are the same as the series!

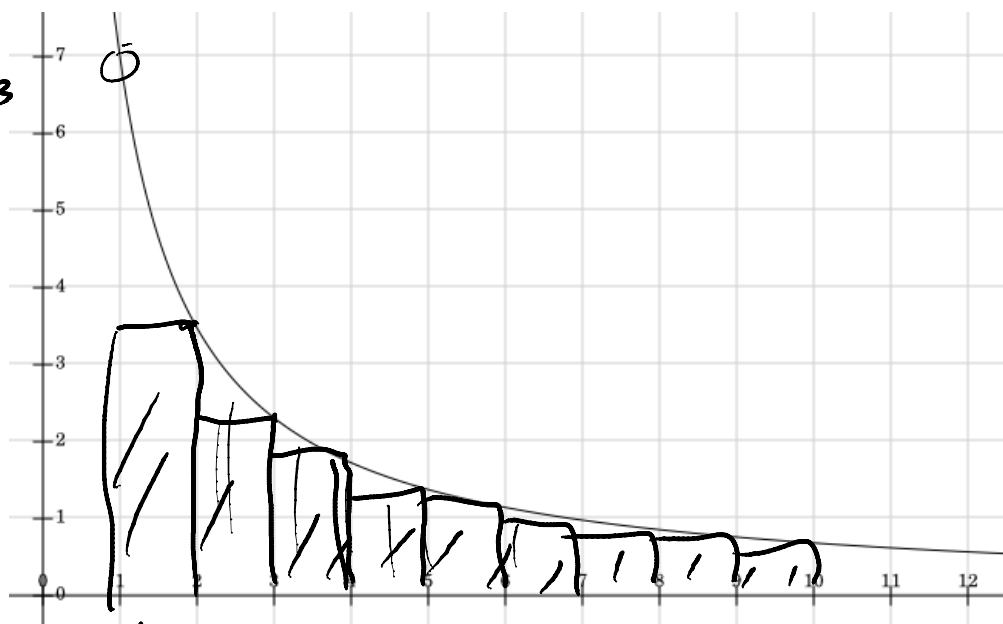
Sec 8.3 Integral and Comparison tests

Lets Suppose we wonder whether or not the series

$$\sum_{i=1}^{\infty} \frac{7}{i^{1.01}} \quad \text{Converges}$$

Converges

$$\rightarrow \frac{7}{1} + \frac{7}{2^{1.01}} + \dots$$



$$\sum_{i=2}^{\infty} \frac{7}{(x_i)^{1.01}} \leq \int_2^{\infty} \frac{7}{x^{1.01}} dx = \int_2^{\infty} 7x^{-1.01} dx$$

$$= 7 \frac{x^{-0.01}}{-0.01} = \frac{-7}{\frac{1}{100}} \frac{1}{x^{0.01}} \Big|_2^{\infty} = -700 \frac{1}{x^{0.01}} \Big|_2^{\infty}$$

$$= -700 \lim_{x \rightarrow \infty} \frac{1}{x^{0.01}} - (-700) \frac{1}{2^{0.01}}$$

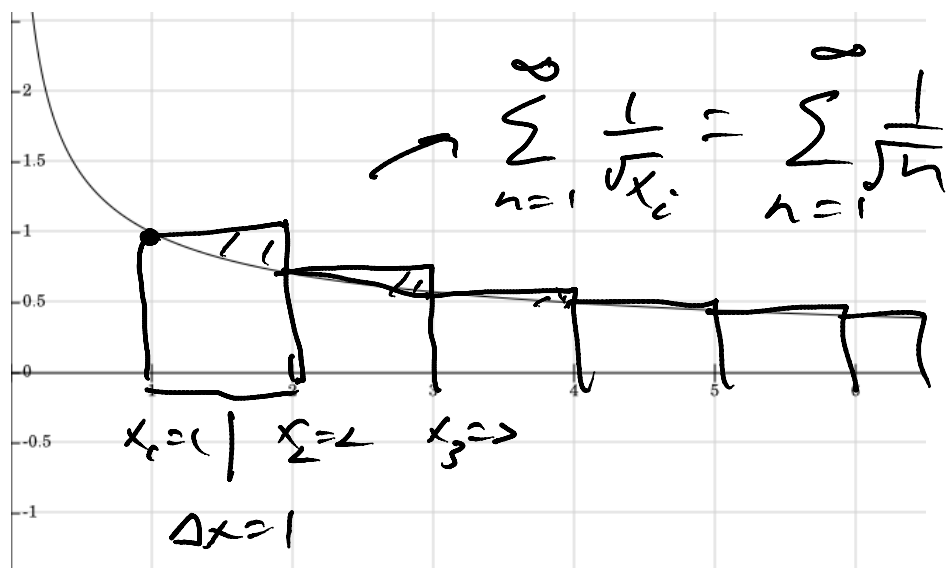
$$= \frac{700}{2^{0.01}}$$

Sec 8.3 Integral and Comparison tests

Does the series converge?

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

$$\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$$



$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{x_i}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \geq \int_1^{\infty} \frac{1}{\sqrt{x}} dx$$

$$\int_1^{\infty} x^{-1/2} dx = \frac{2}{1/2} \left(\lim_{x \rightarrow \infty} x^{1/2} - 1 \right)$$

$$= \frac{2}{1/2} \left(\lim_{x \rightarrow \infty} x^{1/2} - 1 \right) = 4 \left(\lim_{x \rightarrow \infty} x^{1/2} - 1 \right)$$

$$= \infty$$

$$\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \text{ diverges}$$

The Integral Test Suppose f is a continuous, positive, decreasing function on $[1, \infty)$ and let $a_n = f(n)$. Then the series $\sum_{n=1}^{\infty} a_n$ is convergent if and only if the improper integral $\int_1^{\infty} f(x) dx$ is convergent. In other words:

(a) If $\int_1^{\infty} f(x) dx$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent.

(b) If $\int_1^{\infty} f(x) dx$ is divergent, then $\sum_{n=1}^{\infty} a_n$ is divergent.

Ex. Determine whether the series converges

$\sum_{i=1}^{\infty} \frac{1}{n \ln(n)}$

$f(x) = \frac{1}{x \ln(x)}$

$\int_1^{\infty} \frac{1}{x \ln(x)} dx$

$u = \ln(x)$
 $du = \frac{1}{x} dx$
 $dx = x du$

$\int \frac{1}{xu} x du$

$\int \frac{1}{u} du = \ln(u) = \ln(\ln(x)) \Big|_1^{\infty}$

$\lim_{x \rightarrow \infty} \ln(\ln(x)) - \lim_{x \rightarrow 1} \ln(\ln(x))$

$\infty - (-\infty) = \infty$

Sec 8.3 Integral and Comparison tests

Consider the series $\sum_{n=1}^{\infty} \frac{1}{n^p}$

For what values of p does this series converge?

$$a_n = \frac{1}{n^p}$$

$$f(x) = \frac{1}{x^p}$$

$$p \neq 1 \quad \int_1^{\infty} \frac{1}{x^p} dx = \int_1^{\infty} x^{-p} dx = \left. \frac{x^{-p+1}}{-p+1} \right|_1^{\infty}$$

$$\lim_{x \rightarrow \infty} \frac{x^{-p+1}}{-p+1} - \frac{1^{-p+1}}{-p+1}$$

$$\lim_{x \rightarrow \infty} \frac{x^{-p+1}}{-p+1} = \frac{1}{-p+1} \lim_{x \rightarrow \infty} x^{-p+1}$$

$$p < 1$$

$$\lim_{x \rightarrow \infty} x^a \quad a > 0$$

divergence

$$p > 1$$

$$\lim_{x \rightarrow \infty} x^{-a}$$

$$\lim_{x \rightarrow \infty} \frac{1}{x^a} = 0$$

convergence

p -series test

1 The p -series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leq 1$.

The Comparison Test Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms.

(a) If $\sum b_n$ is convergent and $a_n \leq b_n$ for all n , then $\sum a_n$ is also convergent.

(b) If $\sum b_n$ is divergent and $a_n \geq b_n$ for all n , then $\sum a_n$ is also divergent.

a_n, b_n
 $\sum b_n < \infty$ $a_n < b_n \quad \forall n$ $\left| \begin{array}{l} \sum b_n < \infty \\ a_n < b_n \quad \forall n \end{array} \right. \begin{array}{l} \sum a_n < \sum b_n < \infty \\ \sum a_n < \infty \end{array}$

Ex. Show that the series $\sum_{n=0}^{\infty} \frac{1}{n^5 + 3n^2 + 1}$ converges $a_n = \frac{1}{n^5 + 3n^2 + 1}$

$\frac{1}{n^5 + 3n^2 + 1} < \frac{1}{n^5} \quad \forall n$

$\downarrow$ a_n $\downarrow$ b_n $\rightarrow \sum \frac{1}{n^5}$ converges by p-series

$\sum \frac{1}{n^5 + 3n^2 + 1} < \sum \frac{1}{n^5} < \infty$ (Converges!)

$\sum \frac{1}{n^p}$ converges if $p > 1$

Ex. Show that the series $\sum_{n=2}^{\infty} \frac{\ln(n)}{n}$ diverges $a_n = \frac{\ln(n)}{n}$

$\frac{\ln(n)}{n} > \frac{1}{n} \quad \forall n > 1$

$\sum_{n=2}^{\infty} \frac{\ln(n)}{n} > \sum_{n=2}^{\infty} \frac{1}{n} = \infty$ diverges

~~$\frac{\ln(n)}{1} = \frac{0}{1} > \frac{1}{1}$~~

Sec 8.3 Integral and Comparison tests



$\frac{1}{\ln(n)} > \frac{1}{n}$

The Limit Comparison Test Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

$\frac{1}{n} = a_n$
 $\frac{1}{n^2} = b_n$

$\lim_{n \rightarrow \infty} \frac{n^2 - n}{n} = \lim_{n \rightarrow \infty} (n-1) = \infty$

$\sum \frac{1}{\ln(n)}$
 $\frac{1}{n} \cdot \frac{\ln(n)}{1} = \frac{\ln(n)}{n} = \frac{1}{1} = 1 \neq 0$

$\sum \frac{1}{n^2}$ converges
 $\rightarrow a_n$

Example Does the series

$\sum_{n=1}^{\infty} \frac{1}{n^2 - n}$ Converge?
 $\rightarrow b_n$

$\frac{a_n}{b_n} = \frac{\frac{1}{n^2}}{\frac{1}{n^2 - n}} = \frac{1}{n^2} \cdot \frac{n^2 - n}{1} = \frac{n^2 - n}{n^2}$

converges
 ~~$\frac{1}{n^3}$~~

$\lim_{n \rightarrow \infty} \frac{a_n}{b_n} = \lim_{n \rightarrow \infty} \frac{n^2 - n}{n^2} = \frac{1}{1} = 1 > 0$ and finite

What about the series

$\sum_{n=1}^{\infty} \frac{1}{n + \sqrt{n}}$ diverges
 $\rightarrow b_n$

$a_n = \frac{1}{n}$

$\frac{a_n}{b_n} = \frac{\frac{1}{n}}{\frac{1}{n + \sqrt{n}}} = \frac{1}{n} \cdot \frac{n + \sqrt{n}}{1} = \frac{n + \sqrt{n}}{n}$

$\lim_{n \rightarrow \infty} \frac{n + \sqrt{n}}{n} = \lim_{n \rightarrow \infty} \frac{\sqrt{n}(\sqrt{n} + 1)}{n} = \lim_{n \rightarrow \infty} \frac{\sqrt{n} + 1}{\sqrt{n}}$

$= \lim_{n \rightarrow \infty} \frac{\sqrt{n}}{\sqrt{n}} + \frac{1}{\sqrt{n}} = \lim_{n \rightarrow \infty} 1 + \frac{1}{\sqrt{n}} = 1$

Sec 8.3 Integral and Comparison tests

What if we want to know how far off a partial sum is from the full infinite sum? We can define the remainder as:

$$R_N = \sum_{n=1}^{\infty} a_n - \sum_{n=1}^N a_n = \sum_{n=N+1}^{\infty} a_n = s - s_N$$

Then the question is , can we estimate this error?

Let's Think about the integrals.....

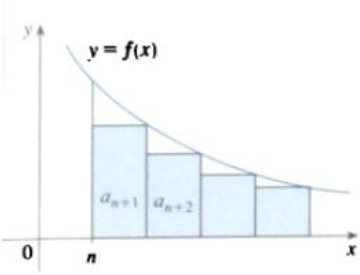


FIGURE 3

$$R_N = \sum_{n=N+1}^{\infty} a_n \leq \int_N^{\infty} a(x) dx$$

Note $\Delta x = 1$

$$\int_{N+1}^{\infty} a(x) dx \leq \sum_{n=N+1}^{\infty} a_n \leq \int_N^{\infty} a(x) dx$$

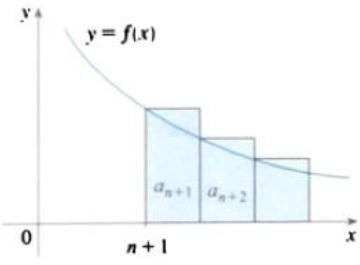


FIGURE 4

$$R_N = \sum_{n=N+1}^{\infty} a_n \geq \int_{N+1}^{\infty} a(x) dx$$

3 Remainder Estimate for the Integral Test Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum a_n$ is convergent. If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x) dx \leq R_n \leq \int_n^{\infty} f(x) dx$$

Sec 8.3 Integral and Comparison tests

Using this we can get bounds on the actual sum

3 Remainder Estimate for the Integral Test Suppose $f(k) = a_k$, where f is a continuous, positive, decreasing function for $x \geq n$ and $\sum a_n$ is convergent. If $R_n = s - s_n$, then

$$\int_{n+1}^{\infty} f(x) dx \leq R_n \leq \int_n^{\infty} f(x) dx$$

Replacing R_n with $s - s_n$ and adding s_n to both sides gives:

$$\int_{n+1}^{\infty} f(x) dx + s_n \leq s \leq \int_n^{\infty} f(x) dx + s_n$$

Bounds on an infinite sum!

$$s_n + \int_{n+1}^{\infty} f(x) dx \leq s \leq s_n + \int_n^{\infty} f(x) dx$$

Ex. Estimate the error in approximating the series below only using 10 terms. $= n = 10$

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\int_{11}^{\infty} \frac{1}{x^2} dx \leq R_n \leq \int_{10}^{\infty} \frac{1}{x^2} dx \Rightarrow \frac{1}{11} \leq R_n \leq \frac{1}{10}$$

$$\int_{11}^{\infty} x^{-2} dx \leq R_n \leq \int_{10}^{\infty} x^{-2} dx \Rightarrow .0909 \leq R_n \leq .1$$

$$\left. \frac{x^{-1}}{-1} \right|_{11}^{\infty} \leq R_n \leq \left. \frac{x^{-1}}{-1} \right|_{10}^{\infty}$$

$$0 - \frac{11^{-1}}{-1} \leq R_n \leq \frac{-10^{-1}}{-1}$$

Sec 8.3 Integral and Comparison tests

How many terms do we need to make sure the error in an estimate of the sum below is within 0.0005?

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

N

$$R_N \leq \int_N^{\infty} \frac{1}{x^2} dx$$

$$R \leq \left. \frac{x^{-1}}{-1} \right|_N^{\infty}$$

$$\leq \left. -\frac{1}{x} \right|_N^{\infty}$$

$$\leq \lim_{x \rightarrow \infty} -\frac{1}{x} - \left(-\frac{1}{N}\right)$$

$$R \leq \frac{1}{N} = .0005$$

$$\frac{1}{N} = .0005 = \frac{5}{10000}$$

$$\frac{1}{N} = \frac{5}{10000} \Rightarrow N = \frac{10000}{5}$$

$$N = 2000$$

Sec 8.3 Integral and Comparison tests

Find bounds for the sum

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$S - S_{10}$$



$$.09 \leq R_{10} \leq .1$$

Using the 10th partial sum

$$\frac{1}{1} + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \frac{1}{7^2} + \frac{1}{8^2} + \frac{1}{9^2} + \frac{1}{10^2}$$

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \frac{1}{49} + \frac{1}{64} + \frac{1}{81} + \frac{1}{100}$$

$$= 1.54976$$

$$1.54976 + .09 \leq S \leq 1.54976 + .1$$

$$1.6406 \leq S \leq 1.64976$$

~~1.64976~~