

Sec 8.7 (Taylor Series!)

Now that we know that we can represent functions with power series it would be nice if we had an algorithm for finding a power series rather than trying to manipulate other known series. Let's Find it!

Suppose $f(x)$ has a power series and is centered at $x=a$ Then we can say that.

$$f(x) = c_0 + c_1(x - a) + c_2(x - a)^2 + c_3(x - a)^3 + c_4(x - a)^4 + c_5(x - a)^5 + \dots$$

As long as $|x-a|<1$ we can plug in an x value and the two sides of the equals sign must be the same.
Suppose we know the function $f(x)$ and lets plug in $x=a$ then

$f(0) = c_0$

So now we know what c_0 must be Next lets differentiate $f(x)$ and its power series and plug in $x=a$

So in general for the power series of $f(x)$ centered at $x=a$ we can say $c_n = \frac{f^n(a)}{n!}$!!!!!

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5 Theorem If f has a power series representation (expansion) at a , that is, if

$$f(x) = \sum_{n=0}^{\infty} c_n(x - a)^n \quad |x - a| < R$$

then its coefficients are given by the formula

$$c_n = \frac{f^{(n)}(a)}{n!}$$

This series is what we call a Taylor series!

6 $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n$

Taylor Series of a function $f(x)$ is given by:

$$= f(a) + \frac{f'(a)}{1!} (x - a) + \frac{f''(a)}{2!} (x - a)^2 + \frac{f'''(a)}{3!} (x - a)^3 + \dots$$

For the Special case when $a=0$ we call the taylor series the Maclaurin Series.

7 $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = f(0) + \frac{f'(0)}{1!} x + \frac{f''(0)}{2!} x^2 + \dots$

What we have now is a way to explicitly calculate a power series for a given function $f(x)$ Of course one very important question is; When do we know for sure that a function will be equal to it's taylor series?

We will get to that but for now lets find some Taylor series.

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$$\boxed{6} \quad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x - a)^n$$
$$= f(a) + \frac{f'(a)}{1!} (x - a) + \frac{f''(a)}{2!} (x - a)^2 + \frac{f'''(a)}{3!} (x - a)^3 + \dots$$

Example Find the Taylor series for e^x centered at $x=0$

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$$\boxed{6} \quad f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$
$$= f(a) + \frac{f'(a)}{1!} (x-a) + \frac{f''(a)}{2!} (x-a)^2 + \frac{f'''(a)}{3!} (x-a)^3 + \dots$$

Find the Taylor series for $\ln(x)$ centered at $x=-1$. For what values of x will this converge? What if I need a value of x not in the interval of convergence?

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When do we know that a function is equal to it's Taylor Series?

Suppose we call the Taylor series of $f(x)$ by the name T When $f(x)$ is equal to its Taylor series $f(x)-T=0$ Simple enough but how can we turn this in to a criteria so that we can know when our Taylor series is equal to it's corresponding function?

Let's think of the partial sum and define $T_N = \sum_{n=0}^N \frac{f^n(a)}{n!} (x - a)^n$ Then we could represent the difference between the Nth taylor polynomial and the function by

$$R_N = f(x) - T_N$$

Of course this means that $f(x) = R_N + T_N$

Note that if we take $\lim_{n \rightarrow \infty} T_N = \lim_{n \rightarrow \infty} \sum_{n=0}^N \frac{f^n(a)}{n!} (x - a)^n = \sum_{n=0}^{\infty} \frac{f^n(a)}{n!} (x - a)^n = T$

If we apply this to our equation for $f(x)$ this says

$$\lim_{n \rightarrow \infty} f(x) = \lim_{n \rightarrow \infty} R_N + \lim_{n \rightarrow \infty} T_N$$

$$\Rightarrow f(x) = \lim_{n \rightarrow \infty} R_N + T$$

Thus only if

$$\lim_{n \rightarrow \infty} R_N = 0$$

Can we say that

$$f(x)=T$$

8 Theorem If $f(x) = T_n(x) + R_n(x)$, where T_n is the n th-degree Taylor polynomial of f at a and

$$\lim_{n \rightarrow \infty} R_n(x) = 0$$

for $|x - a| < R$, then f is equal to the sum of its Taylor series on the interval $|x - a| < R$.

So for a given function $f(x)$ if we can show that $\lim_{n \rightarrow \infty} R_n = 0$ We know it is equal to its Taylor Series. This can be difficult so we make use of what is called Taylor's inequality.....

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9 Taylor's Inequality If $|f^{(n+1)}(x)| \leq M$ for $|x - a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!} |x - a|^{n+1} \quad \text{for } |x - a| \leq d$$

Ex. Find the Taylor series for $\sin(x)$ centered around $x=0$ and prove the series is equal to $\sin(x)$.

To prove this is the Taylor series we need to show that $\lim_{n \rightarrow \infty} R_n = 0$ to do this we first make use of Taylor's inequality...

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Here is a bunch of taylor series! Where are they centered?

$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 +$	$R = 1$
$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} +$	$R = \infty$
$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} +$	$R = \infty$
$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} +$	$R = \infty$
$\tan^{-1}x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} +$	$R = 1$
$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} +$	$R = 1$
$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!} x^2 + \frac{k(k-1)(k-2)}{3!} x^3 +$	$R = 1$

HW Ex #29 Find the taylor series for $f(x) = x\cos(\frac{1}{2}x^2)$

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Use the Taylor series to evaluate the limit

$$\lim_{x \rightarrow 0} \frac{x - \ln(1+x)}{x^2}$$

Evaluate the indefinite integral as a power series HW # 43

$$\int x \cos(x^3) dx$$

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Multiplying or dividing series:

Find the series for $f(x) = \frac{e^x}{1-x}$

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Hw example find the series for $f(x)=x/\sin(x)$ # 57

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An Amazing fact from Taylor Series! Did you know that $e^{i\pi} = -1$ We can show this by considering the Taylor series for e^x