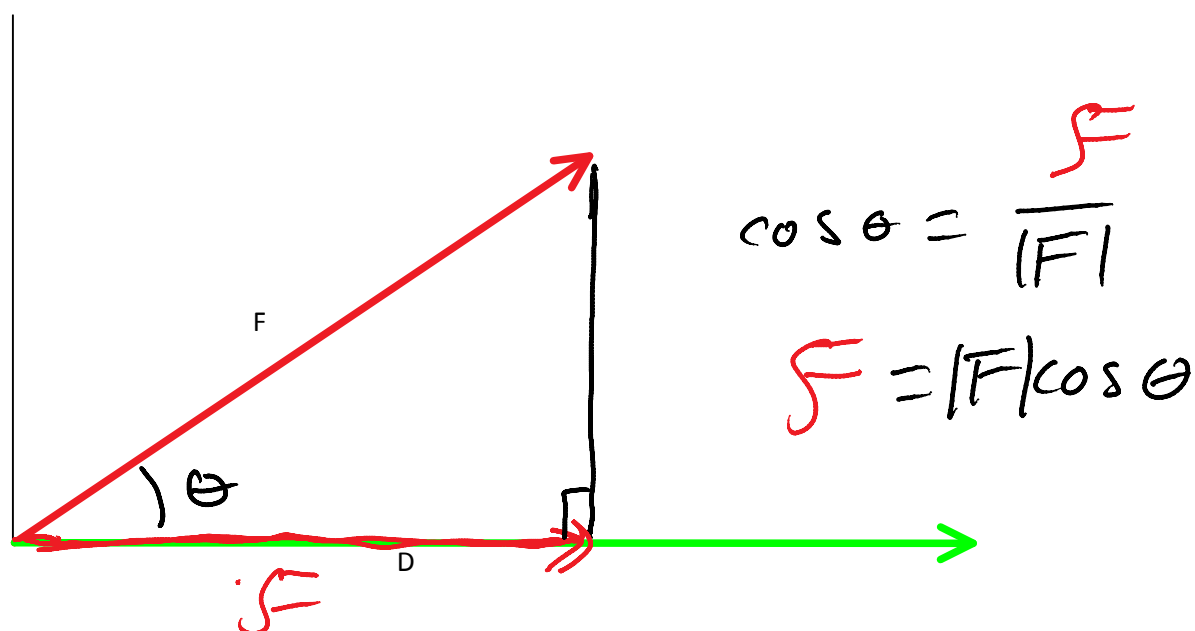


Sections 9.3 and 9.4 Dot and cross product

We have talked about adding and subtracting vectors but we have not yet talked about multiplying them

The Dot Product: The dot product of two vectors comes from the idea of work. Work is only done in the direction of motion and if the force is not in line with the direction of motion we need to find out how much is.



$$W = F \cos \theta |D| = |F| \cos(\theta) |D|$$

$$W = |F| |D| \cos \theta$$

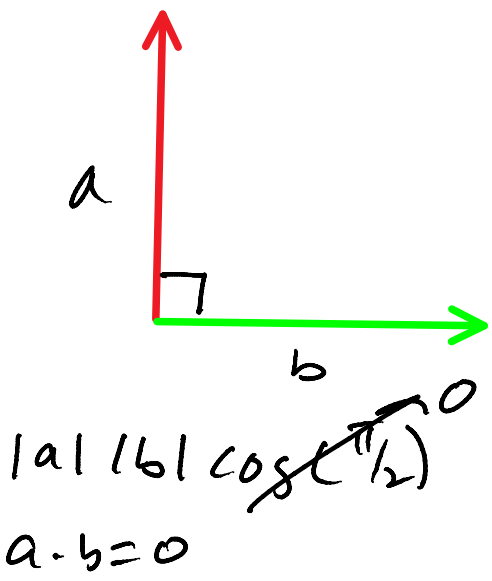
Definition The **dot product** of two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ is the number

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \cos \theta$$

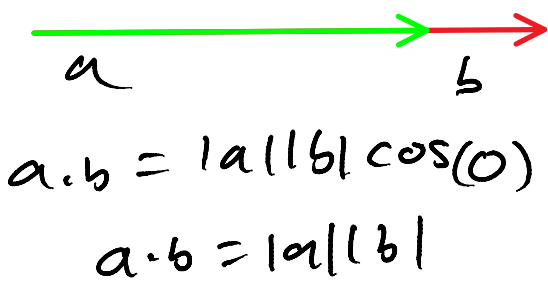
where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, $0 \leq \theta \leq \pi$. (So θ is the smaller angle between the vectors when they are drawn with the same initial point.) If either $\mathbf{a}$ or $\mathbf{b}$ is $\mathbf{0}$, we define $\mathbf{a} \cdot \mathbf{b} = 0$.

Sections 9.3 and 9.4 Dot and cross product

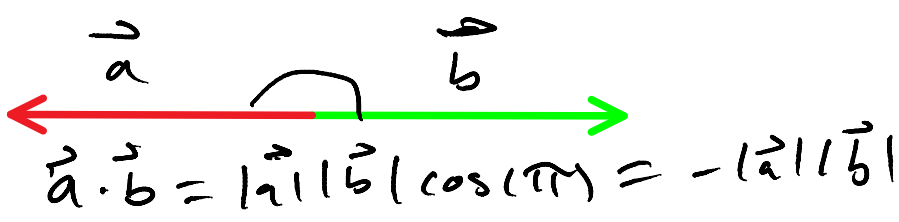
What happens if two vectors are at 90 degrees?



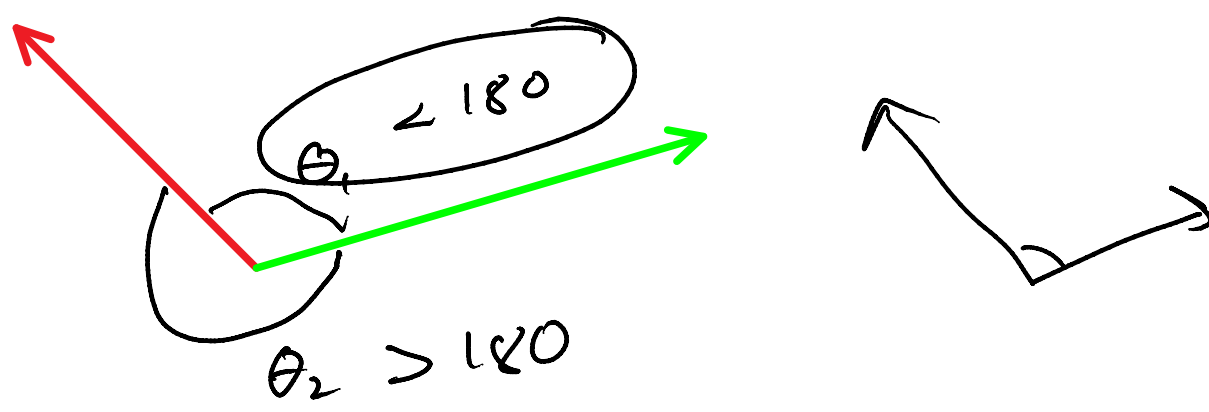
What about 0 degrees



What about 180?



Which angle should you use to calculate a dot product?

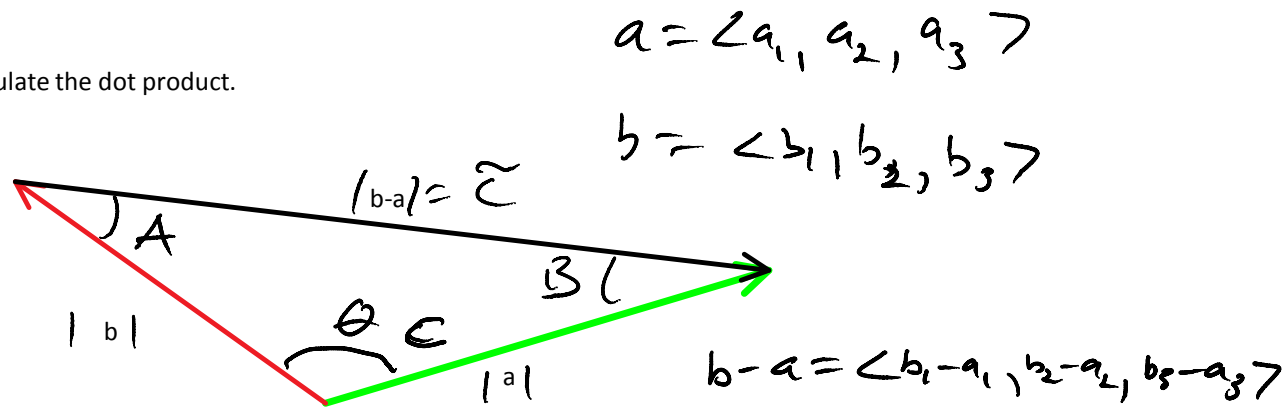


2

Two vectors **a** and **b** are orthogonal if and only if **a · b = 0**.

Sections 9.3 and 9.4 Dot and cross product

Another way to calculate the dot product.



$$c^2 = |a|^2 + |b|^2 - 2|a||b|\cos(\theta)$$

$$|b-a|^2 = |a|^2 + |b|^2 - 2|a||b|\cos(\theta)$$

$a \cdot b$ $\overrightarrow{b-a}$
 $= \langle$

$$2|a||b|\cos\theta = |a|^2 + |b|^2 - |b-a|^2$$

$$2(a \cdot b) = (\sqrt{a_1^2 + a_2^2 + a_3^2})^2 + (\sqrt{b_1^2 + b_2^2 + b_3^2})^2 - (\sqrt{(b_1-a_1)^2 + (b_2-a_2)^2 + (b_3-a_3)^2})^2$$

$$2(a \cdot b) = a_1^2 + a_2^2 + a_3^2 + b_1^2 + b_2^2 + b_3^2 - [(b_1-a_1)^2 + (b_2-a_2)^2 + (b_3-a_3)^2]$$

$$2(a \cdot b) = \cancel{a_1^2} + \cancel{a_2^2} + \cancel{a_3^2} + \cancel{b_1^2} + \cancel{b_2^2} + \cancel{b_3^2} - [\cancel{b_1^2} - 2a_1b_1 + \cancel{a_1^2} + \cancel{b_2^2} - 2a_2b_2 + \cancel{a_2^2} - \cancel{b_3^2} + 2a_3b_3 - \cancel{a_3^2}]$$

$$2(a \cdot b) = 2a_1b_1 + 2a_2b_2 + 2a_3b_3$$

$$a \cdot b = a_1b_1 + a_2b_2 + a_3b_3$$

The dot product of $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ is

$$\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$$

Example Calculate $\langle 1, 2, 3 \rangle \cdot \langle 4, 5, 6 \rangle = 1(4) + 2(5) + 3(6) = 4 + 10 + 18 = \boxed{32}$

Sections 9.3 and 9.4 Dot and cross product

$$a = \langle a_1, a_2, a_3 \rangle \quad b = \langle b_1, b_2, b_3 \rangle \quad c = \langle c_1, c_2, c_3 \rangle$$

Properties of the Dot Product If a , b , and c are vectors in V_3 and c is a scalar, then

1. $a \cdot a = |a|^2$

2. $a \cdot b = b \cdot a$

3. $a \cdot (b + c) = a \cdot b + a \cdot c$

4. $(ca) \cdot b = c(a \cdot b) = a \cdot (cb)$

5. $0 \cdot a = 0$

$$\begin{aligned} \langle a_1, a_2, a_3 \rangle \cdot \langle a_1, a_2, a_3 \rangle &= a_1 \cdot a_1 + a_2 \cdot a_2 + a_3 \cdot a_3 \\ &= a_1^2 + a_2^2 + a_3^2 \end{aligned} \quad |a|^2 = \left(\sqrt{a_1^2 + a_2^2 + a_3^2} \right)^2 = a_1^2 + a_2^2 + a_3^2$$

$$a = \langle 1, 2 \rangle \quad b = \langle 3, 4 \rangle \quad c = \langle 5, 6 \rangle$$

$$a \cdot (b + c) = \langle 1, 2 \rangle \cdot (\langle 3, 4 \rangle + \langle 5, 6 \rangle)$$

$$= \langle 1, 2 \rangle \cdot \langle 8, 10 \rangle = (1)(8) + 2(10) = 8 + 20 = 28$$

$$a \cdot b + a \cdot c = \langle 1, 2 \rangle \cdot \langle 3, 4 \rangle + \langle 1, 2 \rangle \cdot \langle 5, 6 \rangle$$

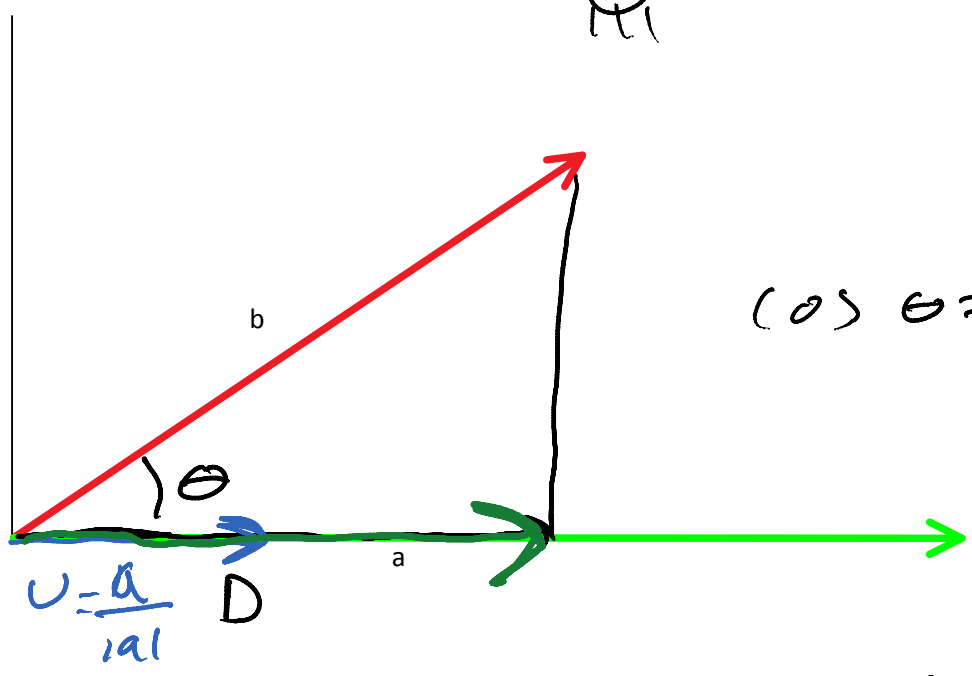
$$= 1(3) + 2(4) + 1(5) + 2(6) = 3 + 8 + 5 + 12$$

$$11 + 17 = \boxed{28}$$

$$2 \langle 1, 2, 3 \rangle = \langle 2, 4, 6 \rangle$$

Sections 9.3 and 9.4 Dot and cross product

Projections : How much of one is in the direction of the other. 



$$\cos \theta = \frac{D}{|b|}$$

$$D = |b| \cos \theta$$

$$\frac{a \cdot b}{|a|} = \frac{|a||b|\cos \theta}{|a|}$$

$$\vec{D} = \frac{a \cdot b}{|a|} \vec{u}$$

$$D = \text{comp}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|a|} \vec{u}$$

$$\vec{D} = \text{proj}_{\vec{a}} \vec{b} = \left(\frac{\vec{a} \cdot \vec{b}}{|\vec{a}|^2} \right) \vec{a}$$

Scalar projection of **b** onto **a**: $\text{comp}_{\vec{a}} \vec{b} = \frac{\vec{a} \cdot \vec{b}}{|a|}$ ← scalar

Vector projection of **b** onto **a**: $\text{proj}_{\vec{a}} \vec{b} = \left(\frac{\vec{a} \cdot \vec{b}}{|a|^2} \right) \vec{a} = \frac{\vec{a} \cdot \vec{b}}{|a|^2} \vec{a}$ ← vector

Sections 9.3 and 9.4 Dot and cross product

1. Which of the following expressions are meaningful? Which are meaningless? Explain.

(a) $(a \cdot b) \cdot c$

(c) $|a| (b \cdot c)$

(e) $a \cdot b + c$

(b) $(a \cdot b)c$

(d) $a \cdot (b + c)$

(f) $|a| \cdot (b + c)$

a, b, c Bold

$\langle 5, 6 \rangle$

$\langle 5, 6, 6 \rangle$

$a = \langle 1, 2 \rangle$ $b = \langle 3, 4 \rangle$

$c = \langle 5, 6 \rangle$

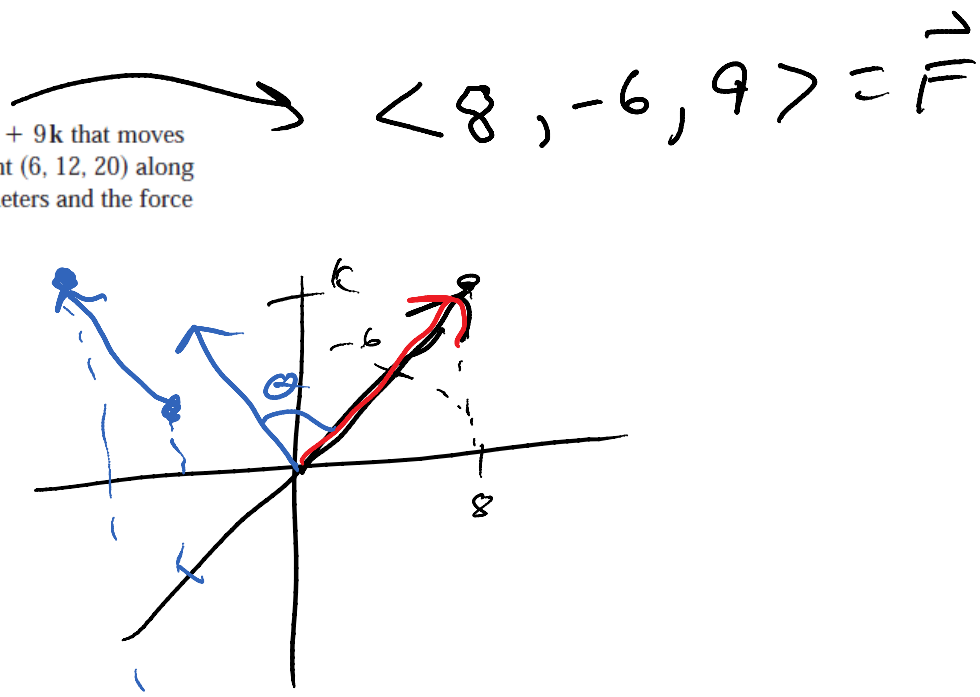
$\langle 1, 2 \rangle \cdot \langle 3, 4 \rangle + \langle 5, 6 \rangle$

$11 + \langle 5, 6 \rangle$

$a \cdot b = 3 + 8 = 11$

$11 \cdot \langle 5, 6 \rangle \neq 11 \langle 5, 6 \rangle$

37. Find the work done by a force $F = 8i - 6j + 9k$ that moves an object from the point $(0, 10, 8)$ to the point $(6, 12, 20)$ along a straight line. The distance is measured in meters and the force in newtons.



$\vec{D} = (6, 12, 20) - (0, 10, 8) = \langle 6-0, 12-10, 20-8 \rangle$

$= \langle 6, 2, 12 \rangle$

$W = |F| |D| \cos \theta = \vec{F} \cdot \vec{D}$

$W = \langle 8, -6, 9 \rangle \cdot \langle 6, 2, 12 \rangle = 8(6) - 6(2) + 9(12)$

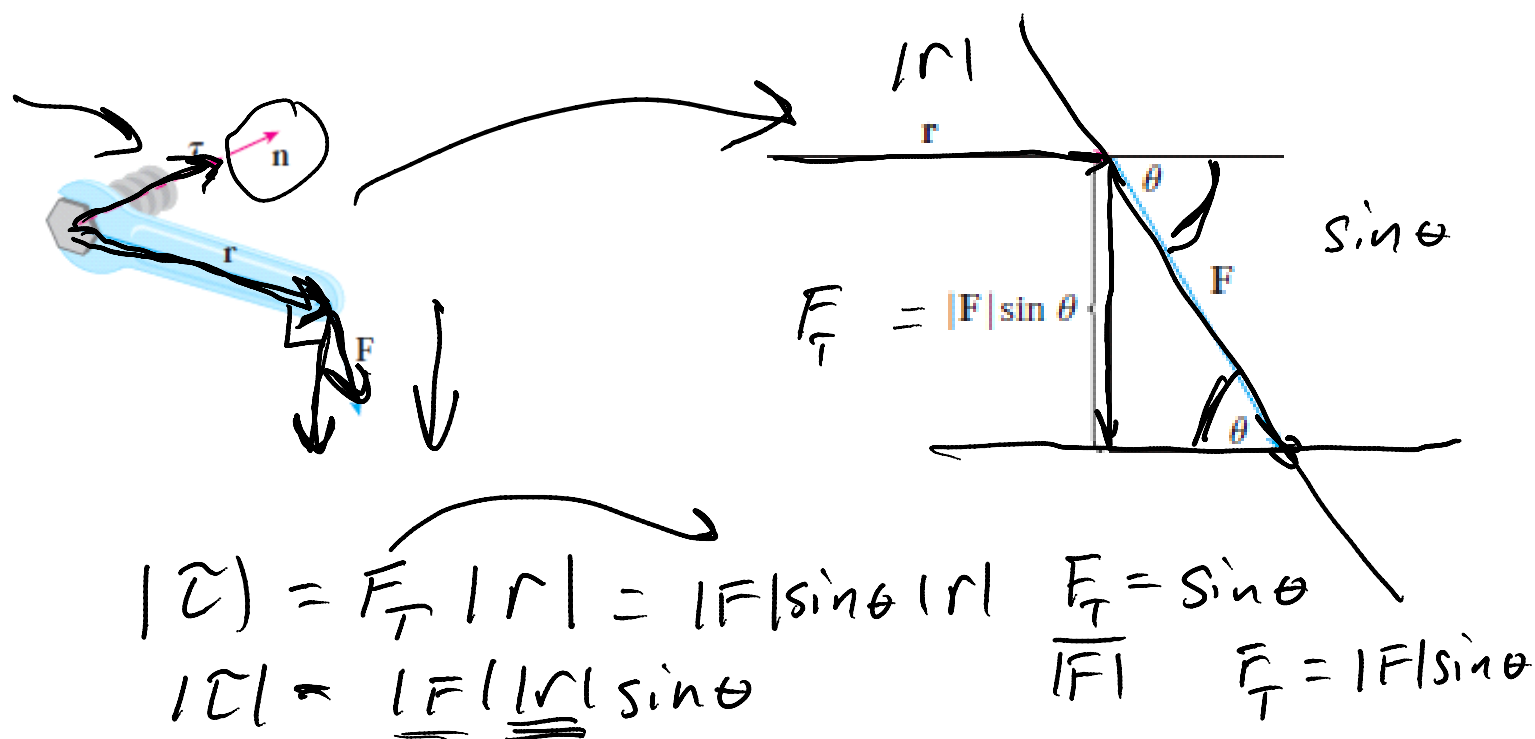
$= 48 - 12 + 108 = \boxed{144 \text{ J}}$

$$W = \angle 8, -6, 1 / \bullet \angle 0, -1, -1 / \dots$$

$$= 48 - 12 + 108 = \boxed{144 \text{ J}}$$

Sections 9.3 and 9.4 Dot and cross product

Another way to multiply vectors is using the cross product. The cross product comes from the idea of torque and is only calculable for 3 dimensional vectors.



Definition If $\mathbf{a}$ and $\mathbf{b}$ are nonzero three-dimensional vectors, the cross product of $\mathbf{a}$ and $\mathbf{b}$ is the vector

$$\mathbf{a} \times \mathbf{b} = (|\mathbf{a}| |\mathbf{b}| \sin \theta) \mathbf{n}$$

$\mathbf{n}$ ← normal vector

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$, $0 \leq \theta \leq \pi$, and $\mathbf{n}$ is a unit vector perpendicular to both $\mathbf{a}$ and $\mathbf{b}$ and whose direction is given by the **right-hand rule**: If the fingers of your right hand curl through the angle θ from $\mathbf{a}$ to $\mathbf{b}$, then your thumb points in the direction of $\mathbf{n}$. (See Figure 3.)

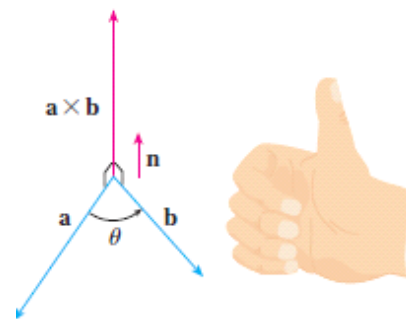
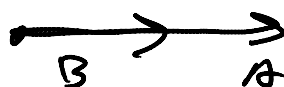


FIGURE 3
The right-hand rule gives the direction of $\mathbf{a} \times \mathbf{b}$.

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= (|\mathbf{a}| |\mathbf{b}| \sin \theta) \mathbf{n} \\ &= (|\mathbf{a}| |\mathbf{b}| \sin 0) \mathbf{n} = \mathbf{0} \end{aligned}$$

$\mathbf{a} \times \mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

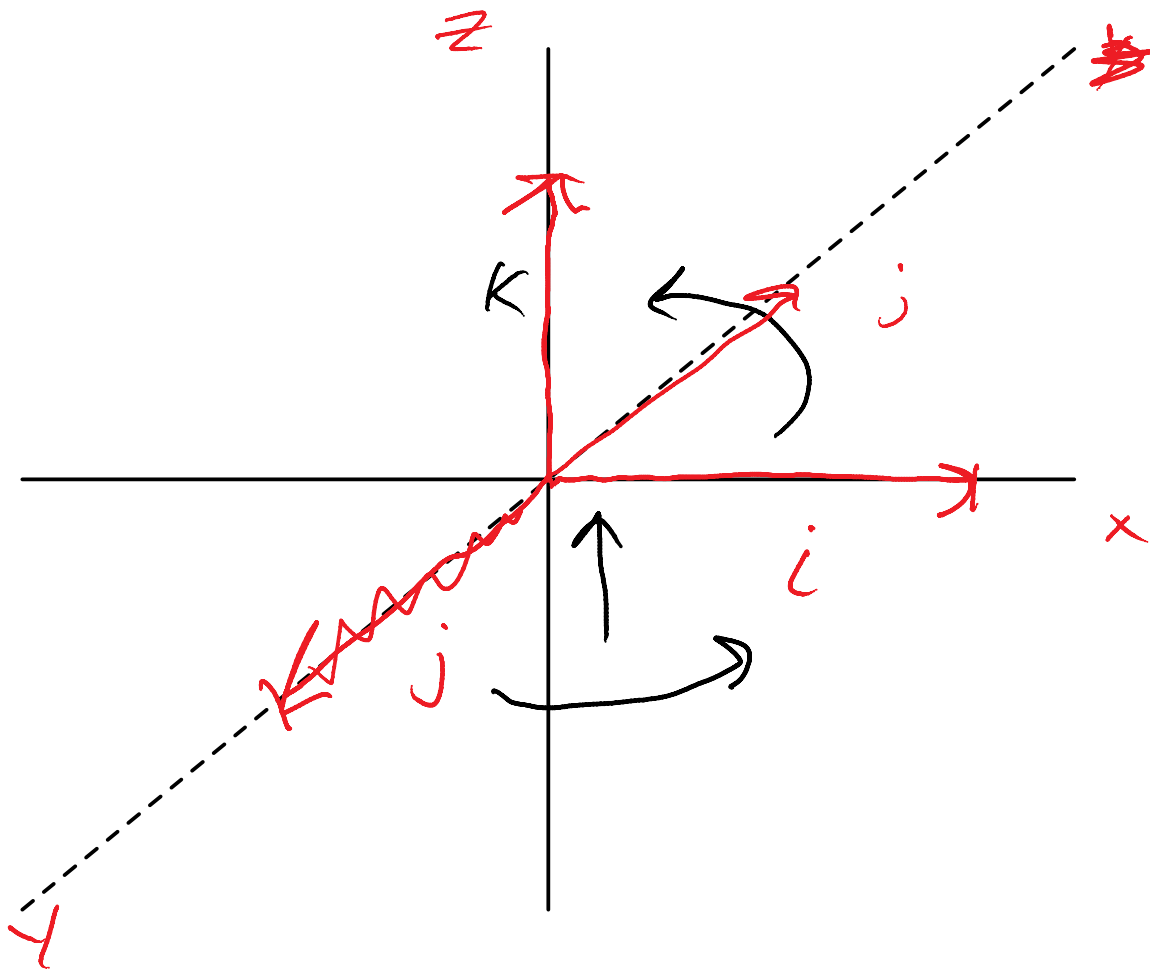


Two nonzero vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if $\mathbf{a} \times \mathbf{b} = \mathbf{0}$. $\rightarrow \langle 0, 0, 0 \rangle$

Sections 9.3 and 9.4 Dot and cross product

Crossing the basis vectors $i=\langle 1,0,0\rangle$ $j=\langle 0,1,0\rangle$ $k=\langle 0,0,1\rangle$

Recall that they are unit vectors and Lets apply the right hand rule.....



$-k = i \times j = (|i||j| \sin(90^\circ)) \vec{n} = (1)(1) \sin(90^\circ) \vec{n} = \vec{n}$

$j \times i = -k$

$i \times k = -j$

$k \times i = j$

$j \times k = i$

$k \times j = -i$

Properties of the Cross Product If **a**, **b**, and **c** are vectors and *c* is a scalar, then

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$

2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$

3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ $\leftarrow$ distributive

4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$ $\leftarrow$

Sections 9.3 and 9.4 Dot and cross product

Properties of the Cross Product If $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors and c is a scalar, then

1. $\mathbf{a} \times \mathbf{b} = -\mathbf{b} \times \mathbf{a}$

2. $(c\mathbf{a}) \times \mathbf{b} = c(\mathbf{a} \times \mathbf{b}) = \mathbf{a} \times (c\mathbf{b})$

3. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$

4. $(\mathbf{a} + \mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$

Another way to calculate the cross product:

Suppose we have $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$. Let's decompose these into a sum of their unit vectors and calculate the cross product.

$\vec{a} = \langle a_1, a_2, a_3 \rangle$

$\vec{a} = a_1 \langle 1, 0, 0 \rangle + a_2 \langle 0, 1, 0 \rangle + a_3 \langle 0, 0, 1 \rangle$

$\vec{a} = a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}$

$\vec{b} = b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k}$

$\vec{a} \times \vec{b} = (a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}) \times (b_1 \mathbf{i} + b_2 \mathbf{j} + b_3 \mathbf{k})$

$(a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}) \times b_1 \mathbf{i} + (a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}) \times b_2 \mathbf{j} + (a_1 \mathbf{i} + a_2 \mathbf{j} + a_3 \mathbf{k}) \times b_3 \mathbf{k}$

$(a_1 \mathbf{i} \times b_1 \mathbf{i}) + (a_2 \mathbf{j} \times b_1 \mathbf{i}) + (a_3 \mathbf{k} \times b_1 \mathbf{i}) + (a_1 \mathbf{i} \times b_2 \mathbf{j}) + (a_2 \mathbf{j} \times b_2 \mathbf{j}) + (a_3 \mathbf{k} \times b_2 \mathbf{j}) + (a_1 \mathbf{i} \times b_3 \mathbf{k}) + (a_2 \mathbf{j} \times b_3 \mathbf{k}) + (a_3 \mathbf{k} \times b_3 \mathbf{k})$

$= a_1 b_1 (\cancel{\mathbf{i} \times \mathbf{i}}) + a_2 b_1 (\mathbf{j} \times \mathbf{i}) + a_3 b_1 (\mathbf{k} \times \mathbf{i}) + a_1 b_2 (\mathbf{i} \times \mathbf{j}) + a_2 b_2 (\cancel{\mathbf{j} \times \mathbf{j}}) + a_3 b_2 (\mathbf{j} \times \mathbf{k}) + a_1 b_3 (\mathbf{i} \times \mathbf{k}) + a_2 b_3 (\mathbf{j} \times \mathbf{k}) + a_3 b_3 (\cancel{\mathbf{k} \times \mathbf{k}})$

$= a_2 b_1 (-\mathbf{k}) + a_3 b_1 (\mathbf{-j}) + a_1 b_2 (\mathbf{k}) + a_3 b_2 (\mathbf{i}) + a_1 b_3 (\mathbf{-j}) + a_2 b_3 (\mathbf{i})$

$+ a_2 b_3 (\mathbf{i}) = (a_3 b_2 - a_2 b_3) \mathbf{i} + (-a_1 b_3 + a_3 b_1) \mathbf{j} + (a_1 b_2 - a_2 b_1) \mathbf{k}$

$\mathbf{i} \times \mathbf{j} = \mathbf{k} \quad \mathbf{j} \times \mathbf{k} = \mathbf{i}$
 $\mathbf{j} \times \mathbf{i} = -\mathbf{k} \quad \mathbf{k} \times \mathbf{j} = -\mathbf{i}$
 $\mathbf{k} \times \mathbf{i} = \mathbf{j} \quad \mathbf{i} \times \mathbf{k} = -\mathbf{j}$

$(a_2b_3 - a_3b_2)i + (a_3b_1 - a_1b_3)j + (a_1b_2 - a_2b_1)k$

2 If $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, then

$$\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1 \rangle$$

5

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

4

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = (a_2b_3 - a_3b_2)\mathbf{i} - (a_1b_3 - a_3b_1)\mathbf{j} + (a_1b_2 - a_2b_1)\mathbf{k}$$

Example Find $\langle 1, 2, 3 \rangle \times \langle 3, 4, 5 \rangle$

$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 3 & 4 & 5 \end{vmatrix}$ $ad - bc$

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 3 \\ 3 & 4 & 5 \end{vmatrix} = \mathbf{i} \begin{vmatrix} 2 & 3 \\ 4 & 5 \end{vmatrix} - \mathbf{j} \begin{vmatrix} 1 & 3 \\ 3 & 5 \end{vmatrix} + \mathbf{k} \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$$

$$\mathbf{i} (2(5) - 4(3)) - \mathbf{j} (5(1) - 3(3)) + \mathbf{k} (1(4) - 3(2))$$

$$-2\mathbf{i} - \mathbf{j}(-4) + \mathbf{k}(-2)$$

$$-2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k} = \langle -2, 4, -2 \rangle = \mathbf{a} \times \mathbf{b}$$

$$x = \frac{ad - bc}{ad - bc} \quad y = \frac{ad - bc}{ad - bc}$$

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

Sections 9.3 and 9.4 Dot and cross product

Scalar triple product.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

6

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$$

7

Triple Cross Product identity

8

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = (\mathbf{a} \cdot \mathbf{c})\mathbf{b} - (\mathbf{a} \cdot \mathbf{b})\mathbf{c}$$

Sections 9.3 and 9.4 Dot and cross product

1. State whether each expression is meaningful. If not, explain why. If so, state whether it is a vector or a scalar.

a, b, c are vectors

- (a) $a \cdot (b \times c) = \text{Scalar}$

(b) ~~$a \times (b \cdot c)$~~
- (c) $a \times (b \times c) = v$

(d) ~~$(a \cdot b) \times c$~~
- (e) ~~$(a \cdot b) \times (c \cdot d)$~~

(f) $(a \times b) \cdot (c \times d) = s$

a)

Calculate $a \times b$ and show it is orthogonal to both a and b

7. $a = \langle 6, 0, -2 \rangle, \quad b = \langle 0, 8, 0 \rangle$

$a \times b = \begin{vmatrix} i & j & k \\ 6 & 0 & -2 \\ 0 & 8 & 0 \end{vmatrix} = i \begin{vmatrix} 0 & -2 \\ 8 & 0 \end{vmatrix} - j \begin{vmatrix} 6 & -2 \\ 0 & 0 \end{vmatrix} + k \begin{vmatrix} 6 & 0 \\ 0 & 8 \end{vmatrix} = +16i + 0j + 48k$

$\hookrightarrow a \times b = \langle 16, 0, 48 \rangle$

$a \cdot (a \times b) = \langle 6, 0, -2 \rangle \cdot \langle 16, 0, 48 \rangle = 96 + 0 - 96 = 0$

$$) \mathbf{a} \times \mathbf{b} = \langle 16, 0, 48 \rangle$$

$$\mathbf{a} = \langle 6, 0, -2 \rangle \quad \mathbf{a} \cdot (\mathbf{a} \times \mathbf{b}) = \langle 6, 0, -2 \rangle \cdot \langle 16, 0, 48 \rangle = 96 + 0 - 96 = 0$$

$$\mathbf{b} = \langle 0, 8, 0 \rangle \quad \mathbf{b} \cdot \mathbf{a} \times \mathbf{b} = \langle 0, 8, 0 \rangle \cdot \langle 16, 0, 48 \rangle$$

$$= 0(16) + 8(0) + 0(48) = 0$$