

Sec 10.4 Motion Through Space

You might recall that the derivative of a position function is velocity and that the derivative of a velocity function is the acceleration. Mathematical arguments aside, this is simply because the rate of change of position with time is velocity and the rate of change of velocity with time is acceleration. In this way we can say:

If the position of a moving object is given by a vector function:

$\vec{r}(t) = \langle f(t), g(t), h(t) \rangle$ $x = f(t)$ $y = g(t)$ $z = h(t)$

Then the velocity of the particle is given by:

$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = \langle f'(t), g'(t), h'(t) \rangle$ The Speed of the particle is therefore given by: $|\vec{v}(t)|$

Its acceleration is given by:

$\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = \frac{d^2\vec{r}(t)}{dt^2} = \langle f''(t), g''(t), h''(t) \rangle$

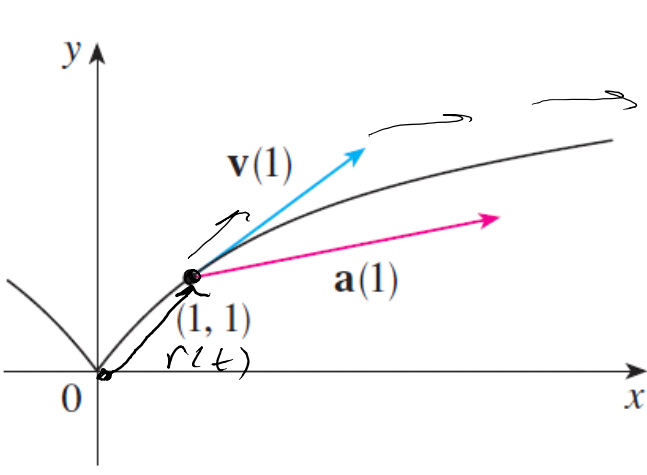
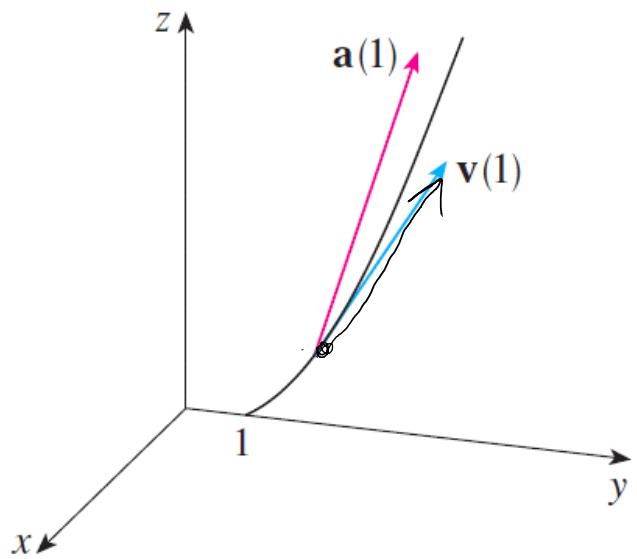


FIGURE 2



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HW Example

3-8 Find the velocity, acceleration, and speed of a particle with the given position function. Sketch the path of the particle and draw the velocity and acceleration vectors for the specified value of t .

7. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + 2\mathbf{k}, \quad t = 1$

$$\vec{r} = \langle t, t^2, 2 \rangle \quad \text{at } t=1 \nearrow \langle 1, 2, 0 \rangle$$

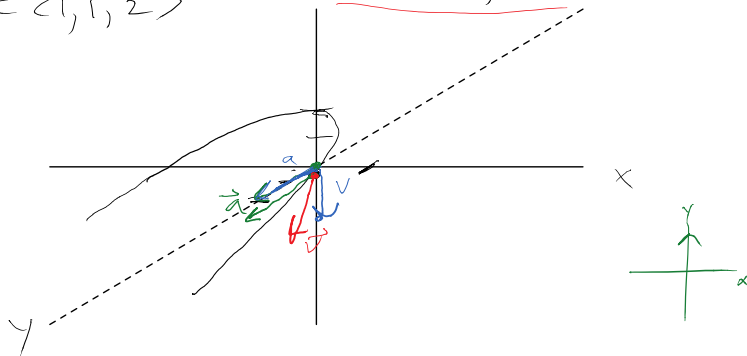
$$\vec{v} = \frac{d}{dt} \vec{r} = \frac{d}{dt} \langle t, t^2, 2 \rangle = \langle 1, 2t, 0 \rangle$$

$$|\vec{v}| = \sqrt{1^2 + (2t)^2 + 0^2} = \sqrt{1 + 4t^2} \quad \text{at } t=1 \quad = \sqrt{1+4} = \sqrt{5}$$

$$\vec{a} = \frac{d}{dt} \vec{v}(t) = \frac{d^2}{dt^2} \vec{r}(t) = \frac{d}{dt} \langle 1, 2t, 0 \rangle$$

$$= \langle 0, 2, 0 \rangle$$

$$\begin{aligned} \vec{r}(t) &= \langle t, t^2, 2 \rangle & \vec{v}(1, 2t, 0) & \vec{a} = \langle 0, 2, 0 \rangle \\ \vec{r}(1) &= \langle 1, 1, 2 \rangle & \vec{v}(1) &= \langle 1, 2, 0 \rangle \end{aligned}$$



$$\vec{r}(t) = \langle t, t^2, 2 \rangle$$

$\langle t, t^2 \rangle$

$x = t$
 $y = t^2$
 $y = x^2$

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Uniform Circular Motion

$$\cos(\omega t)$$

$$\frac{2\pi}{\omega} = \frac{2\pi}{\omega}$$

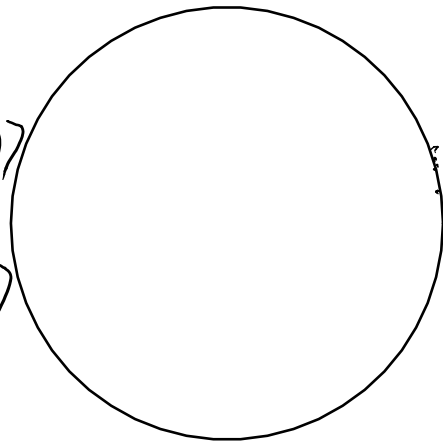
Suppose we have a particle moving in a circle with position vector given by $\vec{R}(t) = \langle r \cos(\omega t), r \sin(\omega t) \rangle$

What is the force acting on the particle?

$$\vec{F} = m \vec{a}$$

$$\frac{d}{dt} \vec{R}(t) = \vec{V}(t) = \langle -r\omega \sin \omega t, r\omega \cos \omega t \rangle$$

$$\vec{a} = \frac{d}{dt} \vec{V}(t) = \langle -r\omega^2 \cos(\omega t), -r\omega^2 \sin(\omega t) \rangle$$

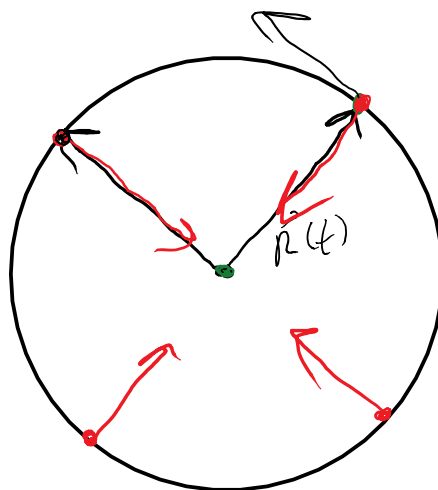


$$\vec{F} = m \vec{a}$$

$$\vec{F} = m \langle -r\omega^2 \cos(\omega t), -r\omega^2 \sin(\omega t) \rangle$$

$$\vec{F} = -m\omega^2 \langle r \cos(\omega t), r \sin(\omega t) \rangle$$

$$\vec{F} = -m\omega^2 \vec{R}(t)$$



15-16

(a) Find the position vector of a particle that has the given acceleration and the specified initial velocity and position.

(b) Use a computer to graph the path of the particle.

15. $\mathbf{a}(t) = 2t\mathbf{i} + \sin t\mathbf{j} + \cos 2t\mathbf{k}$, $\mathbf{v}(0) = \mathbf{i}$, $\mathbf{r}(0) = \mathbf{j}$

$$\vec{a}(t) = \langle 2t, \sin t, \cos(2t) \rangle \quad \begin{array}{l} \boxed{\mathbf{v}(0) = \langle 1, 0, 0 \rangle} \\ \boxed{\mathbf{r}(0) = \langle 0, 1, 0 \rangle} \end{array}$$

$$\int \vec{a}(t) dt = \langle \int 2t dt, \int \sin(t) dt, \int \cos(2t) dt \rangle$$

$$\vec{v}(t) = \langle \frac{2}{2}t^2 + C_1, -\cos(t) + C_2, \frac{1}{2}\sin(2t) + C_3 \rangle$$

$$\mathbf{v}(0) = \langle 1, 0, 0 \rangle = \langle \frac{2}{2}0^2 + C_1, -\cos(0) + C_2, \frac{1}{2}\sin(2 \cdot 0) + C_3 \rangle$$

$$= \langle 1, 0, 0 \rangle = \langle C_1, -1 + C_2, 0 + C_3 \rangle$$

$$\Rightarrow C_1 = 1 \quad 0 = -1 + C_2 \quad 0 = C_3$$

$$\Rightarrow C_2 = 1$$

$$\vec{v}(t) = \langle t^2 + 1, -\cos(t) + 1, \frac{1}{2}\sin(2t) \rangle$$

$$\vec{r}(t) = \langle \int t^2 + 1, \int -\cos(t) + 1, \int \frac{1}{2}\sin 2t \rangle$$

$$\int \sin(kt) = -\frac{1}{k} \cos(kt)$$

$$\int e^{Bt} = \frac{1}{B} e^{Bt}$$

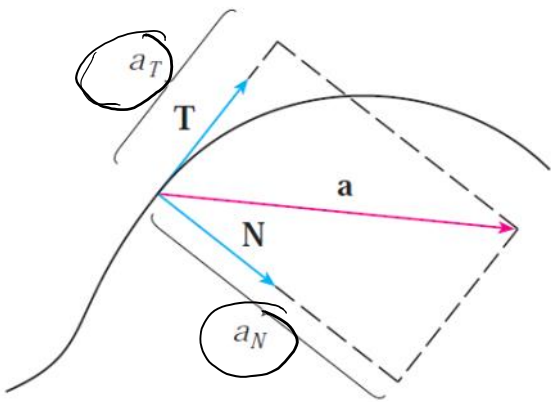
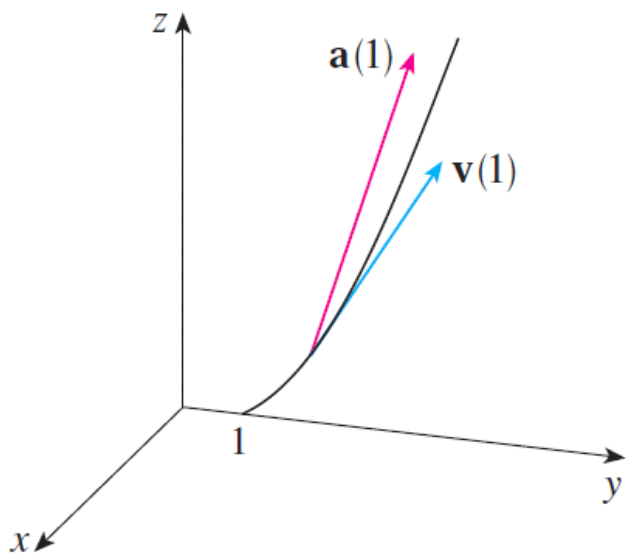
$$\int \cos(2t) dt \quad \begin{array}{l} \text{let } u = 2t \\ du = 2 dt \end{array}$$

$$\int \cos(u) \frac{du}{2} \quad dt = \left(\frac{du}{2} \right)$$

$$\frac{1}{2} \int \cos(u) du$$

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Tangential and Normal components of acceleration.



$$\vec{a} = a_T \vec{T} + a_N \vec{N}$$

What are a_T and a_N ? How do we find them? We need to decompose the acceleration vector into a sum of two orthogonal vectors.....

We can do this using our definitions of the Tangential and Normal vectors and Curvature.
And recalling this rule for scalar times vector function differentiation

3. $\frac{d}{dt} [f(t)\mathbf{u}(t)] = f'(t)\mathbf{u}(t) + f(t)\mathbf{u}'(t)$

$$\vec{a} = a_T \vec{T} + a_N \vec{N}$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\mathbf{v}(t)}{|\mathbf{v}(t)|} = \frac{\mathbf{v}}{v}$$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$$

$$\kappa = \frac{|\mathbf{T}'|}{|\mathbf{r}'|} = \frac{|\mathbf{T}'|}{v} \quad \text{so} \quad |\mathbf{T}'| = \kappa v$$

$$\vec{v} = v\vec{T}$$

$$v\kappa = \frac{|\mathbf{T}'|}{v} \quad \text{so} \quad \kappa v = |\mathbf{T}'|$$

$$\frac{d}{dt} \vec{v} = \frac{d}{dt} v \vec{T}$$

$$\vec{N} = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} = \frac{\vec{T}'}{\kappa v}$$

$$\vec{a}(t) = v' \vec{T} + v \vec{T}'$$

$$\vec{N} = \frac{\vec{T}'}{\kappa v}$$

$$\vec{a}(t) = a_T \vec{T} + a_N \vec{N}$$

$$\vec{T}' = \kappa v \vec{N}$$

$$\vec{a}(t) = v' \vec{T} + \kappa v^2 \vec{N}$$

$$\vec{a}(t) = \underbrace{a_T}_{\text{red}} \vec{T} + \underbrace{a_N}_{\text{red}} \vec{N}$$

$$\vec{a}(t) = v' \vec{T} + kv^2 \vec{N}$$

$$\vec{T} = \frac{d\vec{r}}{ds} = \frac{d\vec{r}}{dt} \frac{dt}{ds} = \frac{v}{k} \vec{T}$$

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So if we decompose the acceleration vector into a sum of two orthogonal vectors we have:

$$\mathbf{a} = v' \mathbf{T} + \kappa v^2 \mathbf{N}$$

Where $v = |\mathbf{v}(t)|$ is the speed, $\mathbf{T}$ the unit tangent vector, κ the curvature and $\mathbf{N}$ the unit normal vector.

We can also write this as:

$$\mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N}$$

$$a_T = v'$$

and

$$a_N = \kappa v^2$$

One can also show that:

$$a_T = v' = \frac{\mathbf{v} \cdot \mathbf{a}}{v} = \frac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|}$$

and

$$a_N = \kappa v^2 = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} |\mathbf{r}'(t)|^2 = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|}$$

$$\mathbf{a} = a_T \mathbf{T} + a_N \mathbf{N}$$

HW Example

33-36 Find the tangential and normal components of the acceleration vector.

$$a_T = v'$$

and

$$a_N = \kappa v^2$$

35. $\mathbf{r}(t) = \cos t \mathbf{i} + \sin t \mathbf{j} + t \mathbf{k} = \langle \cos t, \sin t, t \rangle$

$$a_T = v'$$

$$\vec{v}(t) = \frac{d}{dt} \vec{r}(t) = \langle -\sin t, \cos t, 1 \rangle$$

$$|\vec{v}(t)| = v = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{1+1} = \sqrt{2}$$

$$a_T = \frac{d}{dt} v = v' = \frac{d}{dt} \sqrt{2} = 0$$

$$a_N = \kappa v^2 = \kappa (\sqrt{2})^2 = 2\kappa$$

$$\kappa = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{|\mathbf{T}'(t)|}{v} \leftarrow$$

$$\mathbf{T}(t) = \frac{\vec{v}(t)}{|\vec{v}(t)|} = \frac{\vec{v}(t)}{v} = \langle \frac{-\sin t}{\sqrt{2}}, \frac{\cos t}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$$

$$\mathbf{T}'(t) = \langle \frac{-\cos t}{\sqrt{2}}, \frac{-\sin t}{\sqrt{2}}, 0 \rangle \quad \left(\frac{-\cos t}{\sqrt{2}} \right)^2 = \frac{\cos^2 t}{2}$$

$$|\mathbf{T}'(t)| = \sqrt{\frac{\cos^2 t}{2} + \frac{\sin^2 t}{2} + 0^2} = \sqrt{\frac{1}{2}(\cos^2 t + \sin^2 t)} = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

$$\kappa = \frac{1/\sqrt{2}}{\sqrt{2}} = \frac{1}{2} \quad \Rightarrow \quad a_N = \kappa v^2 = \frac{1}{2} (\sqrt{2})^2 = \frac{1}{2} \cdot 2 = 1$$

$$\vec{a}(t) = 0 \vec{T} + (1) \vec{N} \Rightarrow \vec{a}(t) = \vec{N}$$

$$\frac{2}{\sqrt{2}} = \frac{\sqrt{2} \sqrt{2}}{\sqrt{2}}$$

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Back to that circular motion..... We found that the acceleration on a particle traveling in a circle was

given by: $\vec{a}(t) = -\omega^2 \langle r \cos(\omega t), r \sin(\omega t) \rangle = -\omega^2 \vec{R}(t)$

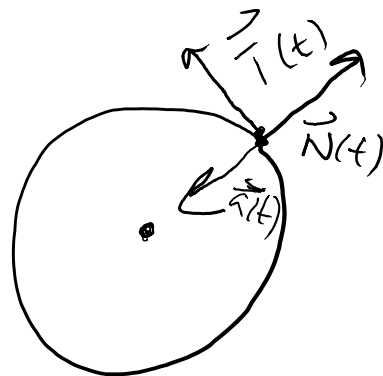
$$\vec{a}(t) = -\omega^2 \langle r \cos(\omega t), r \sin(\omega t) \rangle$$

$$\vec{a}(t) = -\omega^2 \vec{R}(t)$$

Let's decompose this into the normal and tangential components... We know:

$$\vec{R}(t) = \langle r \cos(\omega t), r \sin(\omega t) \rangle$$

$$\mathbf{a} = v' \mathbf{T} + \kappa v^2 \mathbf{N}$$



$$\vec{a}(t) = -\omega^2 \vec{R}(t)$$

$$-\omega^2 \vec{R}(t) = \cancel{v' \mathbf{T}} + \kappa v^2 \vec{N}$$

$$-\omega^2 \vec{R}(t) = \underline{\kappa} v^2 \vec{N}$$

$$\kappa = \frac{1}{r}$$

$$\vec{a}(t) = -\omega^2 \vec{R}(t) = \frac{1}{r} v^2 \vec{N}$$

$$|\vec{a}(t)| = |-\omega^2 \vec{R}(t)| = \left| \frac{1}{r} v^2 \vec{N} \right|$$

$$|\vec{a}(t)| = \omega^2 |\vec{R}(t)| = \frac{1}{r} v^2 |\vec{N}|$$

$$= \omega^2 r = \frac{1}{r} v^2 = \omega^2 r = \frac{v^2}{r} =$$

$$a = \omega^2 r \quad a = \frac{v^2}{r}$$

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Back to that roller coaster... Recall from the last lab we had a roller coaster moving down a spiral section of a track with position vector $\vec{r} = \langle 10 \sin(\pi t), 10 \cos(\pi t), 100 - 10t \rangle$. Find The force on the coster car as it moves down the track.